COMPACTLY SUPPORTED WAVELETS AND REPRESENTATIONS OF THE CUNTZ RELATIONS

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ABSTRACT. We study the harmonic analysis of the quadrature mirror filters coming from multiresolution wavelet analysis of compactly supported wavelets. It is known that those of these wavelets that come from third order polynomials are parametrized by the circle, and we compute that the corresponding filters generate irreducible mutually disjoint representations of of the Cuntz algebra $\mathcal{O}_2$ except at two points on the circle. One of the two exceptional points corresponds to the Haar wavelet and the other is the unique point on the circle where the father function defines a tight frame which is not an orthonormal basis. At these two points the representation decomposes into two and three mutually disjoint irreducible representations, respectively, and the two representations at the Haar point are each unitarily equivalent to one of the three representations at the other singular point.

1. INTRODUCTION

In this paper we show that wavelets may be constructed from representations of two systems of operator relations, one on $L^2(\mathbb{R})$ and one on $L^2(\mathbb{T})$, for the case of one real dimension. Focusing on the case of compact support, the analysis reduces to a certain finite-dimensional matrix problem which is especially amenable to an algorithmic and computational approach. The associated algorithms are worked out in detail for a variety of examples which includes the Daubechies wavelet, and which also reveals some perhaps unexpected symmetries.

One benefit from the representation theoretic approach to wavelets is that it provides a coordinate-free way of making precise notions of irreducibility which occur in the wavelet literature without always having precise definitions. Specifically, examples in $L^2(\mathbb{R}^d)$, for $d > 1$, may occasionally be reduced to simpler examples in one dimension, i.e., in $L^2(\mathbb{R})$, by a tensor product construction, but this analysis depends on the chosen spatial coordinates in $\mathbb{R}^d$, while the representation-theoretic approach in the present paper does not.

One of our results, Corollary 3.3, specifies in a general context (for compactly supported wavelets in $\mathbb{R}^\nu$) a decomposition formula (finite orthogonal sums of irreducible representations) for the representation associated with a system of high-pass/low-pass filters which generate the wavelets in question.

It has been known for some time that a class of convolution operators from signal analysis, called subband filters, satisfy certain operator relations [31, Lemma 2.1]. Perhaps it is less well known among experts in multiresolution wavelet theory

1991 *Mathematics Subject Classification.* Primary 46L60, 47D25, 42A16, 43A65; Secondary 46L45, 42A65, 41A15.

Key words and phrases. Wavelet, Cuntz algebra, representation, orthogonal expansion, quadrature mirror filter, isometry in Hilbert space.

Research supported by the University of Oslo.

that these operator relations were introduced in C^*-algebra theory by J. Dixmier [14, Exemple 2.1] and J. Cuntz [10] several decades ago, and the C^*-algebra they generate is now called the Cuntz algebra of order N and is denoted by $\mathcal{O}_N$, where N is the scale of the resolution. This algebra is independent of the particular scale-N multiresolution wavelet, but the unitary equivalence class of the corresponding representation may depend on the wavelet. The detailed structure of these representations has, however, so far only been worked out in the single case of the Haar wavelet (see below). The purpose of the present paper is to work out the structure of these representations for all compactly supported wavelets, using a method tailor-made for the purpose in [6]. We will show that all representations obtained from compactly supported wavelets have a finite-dimensional commutant, and as a consequence they decompose into a finite direct sum of irreducible representations. We also display a one-parameter family (with two singular points) of mutually inequivalent representations of $\mathcal{O}_2$ on $L^2(\mathbb{T})$ for which the corresponding family of wavelets contains Daubechies's continuous, one-sided differentiable mother function, $\psi \in L^2(\mathbb{R})$, supported on $[0,3] \subset \mathbb{R}$. In our one-parameter family of wavelets supported on $[0,3]$, there is actually a left-handed and a paired right-handed Daubechies wavelet, resulting from a natural symmetry in the family. In going from one to the other, the one-sided differentiability property reverses direction.

Let us briefly review how one constructs representations from a multiresolution wavelet of scale N. Many more details may be found in [5]. Excellent accounts of multiresolution wavelet analysis in general may be found in [21] and [9].

Define scaling by N on $L^2(\mathbb{R})$ as the unitary operator U given by $(U\xi)(x) = N^{-\frac{1}{2}}\xi\left(N^{-1}x\right)$ for $\xi \in L^2(\mathbb{R})$, $x \in \mathbb{R}$, and translation as the unitary operator T given by $(T\xi)(x) = \xi(x-1)$. There is a *father function* or *scaling function* φ which is a vector in $L^2(\mathbb{R})$ such that

$$\left\{T^k\varphi\right\}_{k\in\mathbb{Z}} \text{ is an orthonormal set in } L^2(\mathbb{R}). \tag{1.1}$$

Furthermore, one assumes that there is a sequence $(b_n) \in \ell^2$ such that

$$U\varphi = \sum_n b_n T^n \varphi, \tag{1.2}$$

and then necessarily $\sum_n |b_n|^2 = 1$. (It seems to be fairly conventional in wavelet theory to only consider real b, but this is not too important for what follows.) A weaker, so-called "tight frame", property for the vectors in (1.1) will also be considered as a degenerate case in Section 4.1.2. If $\mathcal{V}_0$ is the closed subspace of $L^2(\mathbb{R})$ spanned by $\left\{T^k\varphi\right\}_{k\in\mathbb{Z}}$, one also assumes

$$\bigwedge_{n\in\mathbb{Z}} U^n\mathcal{V}_0 = \{0\}, \qquad \bigvee_{n\in\mathbb{Z}} U^n\mathcal{V}_0 = L^2(\mathbb{R}). \tag{1.3}$$

These are all the properties of the father function φ that are needed. One example is the Haar father function $\varphi(x) = \chi_{[0,1]}(x)$.

Define a function m_0 in $L^2(\mathbb{T})$ by

$$m_0(t) = m_0\left(e^{-it}\right) = \sum_n b_n e^{-int}. \tag{1.4}$$

Choose functions $m_1, \dots, m_{N-1}$ in $L^2(\mathbb{T})$ such that

$$\sum_{k=0}^{N-1} \overline{m_i\left(t+\frac{2\pi k}{N}\right)} m_j\left(t+\frac{2\pi k}{N}\right) = \delta_{ij} N \tag{1.5}$$

for almost all $t \in \mathbb{R}$, $i, j = 0, 1, \dots, N-1$, or, equivalently, such that the $N \times N$ matrix

$$\frac{1}{\sqrt{N}} \begin{pmatrix} m_0(z) & m_0(\rho z) & \dots & m_0(\rho^{N-1}z) \\ m_1(z) & m_1(\rho z) & \dots & m_1(\rho^{N-1}z) \\ \vdots & \vdots & \ddots & \vdots \\ m_{N-1}(z) & m_{N-1}(\rho z) & \dots & m_{N-1}(\rho^{N-1}z) \end{pmatrix}, \tag{1.6}$$

where $\rho = e^{\frac{2\pi i}{N}}$, is unitary for almost all $z \in \mathbb{T}$. (With m_0 given as above, $m_1, \dots, m_{N-1}$ may always be so chosen; see, e.g., [5].) If we define $\psi_1, \dots, \psi_{N-1} \in L^2(\mathbb{R})$ by

$$\sqrt{N}\hat{\psi}_i(Nt) = m_i(t)\,\hat{\varphi}(t) \tag{1.7}$$

for $t \in \mathbb{R}$, $i = 1, \dots, N-1$, where $\hat{}$ denotes Fourier transform, unitarity of the above matrix is equivalent to orthonormality in $L^2(\mathbb{R})$ of the set

$$\left\{U^n T^k \psi_i\right\}_{n,k\in\mathbb{Z};\, i=1,\dots,N-1}. \tag{1.8}$$

The ψ_i's are called the mother functions. If $N = 2$, there is only one, of course.

Unitarity of (1.6) is also equivalent to saying that the operators S_i, defined on $L^2(\mathbb{T})$ by

$$(S_i\xi)(z) = m_i(z)\,\xi\left(z^N\right) \tag{1.9}$$

for $\xi \in L^2(\mathbb{T})$, $z \in \mathbb{T}$, $i = 0, 1, \dots, N-1$, satisfy the relations

$$S_j^* S_i = \delta_{ij}\mathbb{1}, \qquad \sum_{i=0}^{N-1} S_i S_i^* = \mathbb{1}, \tag{1.10}$$

which are exactly the Cuntz relations. There is a one-to-one correspondence between operator solutions to (1.10) and representations of $\mathcal{O}_N$, and since $\mathcal{O}_N$ is simple, these representations are always faithful. The Fourier transform of S_i^* (the adjoint of (1.9)), acting on $\ell^2(\mathbb{Z})$, is the quadrature mirror filter F_i in [31]: F_0 is low-pass, and $F_1, \dots, F_{N-1}$ are the corresponding high-pass filters for the signal reconstitution process. Let

$$m_i(z) = \sum_n a_n^{(i)} z^n \tag{1.11}$$

be the Fourier decomposition. It follows from (1.9) that for $x = (x_k)_{k\in\mathbb{Z}} \in \ell^2$, we have

$$\left(F_j^* x\right)_n = \sum_{k\in\mathbb{Z}} a_{n-Nk}^{(j)} x_k, \qquad (F_j x)_n = \sum_{k\in\mathbb{Z}} \overline{a_{k-Nn}^{(j)}} x_k, \tag{1.12}$$

as operators $\ell^2 \to \ell^2$. The Cuntz relations in ℓ^2-operator form,

$$F_i F_j^* = \delta_{ij}\mathbb{1}, \qquad \sum_{j=0}^{N-1} F_j^* F_j = \mathbb{1}, \tag{1.13}$$

then summarize subband filtering, which can be written in diagram form as in Figure 1. Here "analysis" is splitting into subbands and the application of F_i, and

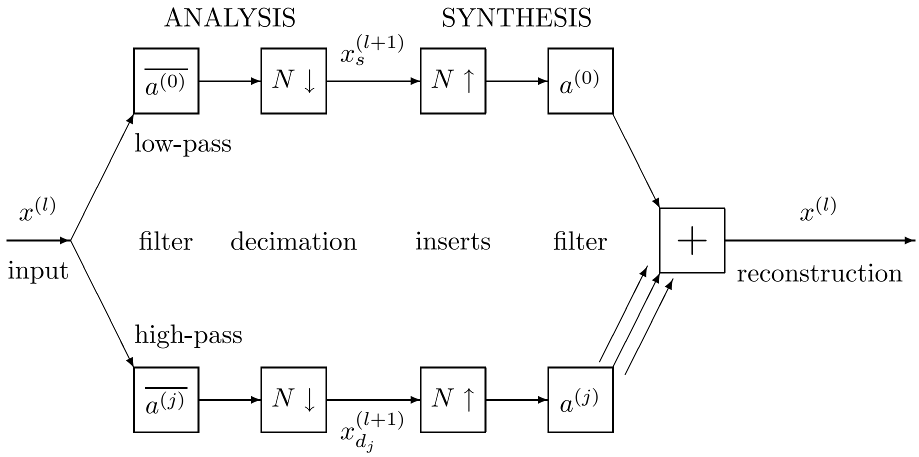


FIGURE 1. Signal subband filtering

"synthesis" is the application of F_i^* followed by summing over the subbands again. The low-pass subband corresponds to $i = 0$, and the high-pass subbands correspond to $i = 1, \dots, N-1$. See [31] and [9] for details.

The $\mathcal{O}_N$-representations given in (1.9) play a crucial role in the wavelet analysis in a second related way. A scale-N wavelet in $L^2(\mathbb{R})$ is an orthonormal basis (or a tight frame) of the form (1.8) as described above. An important point is that the corresponding S_i-operators of (1.9), which constitute the $\mathcal{O}_N$-representation, enter directly and explicitly into a formula for the $L^2(\mathbb{R})$-expansion coefficients c_{nki} of $\xi = \sum_{n,k,i} c_{nki}(\xi)\, U^n T^k \psi_i$, $\xi \in L^2(\mathbb{R})$, and we refer to [5, eq. (1.35)] for details on that.

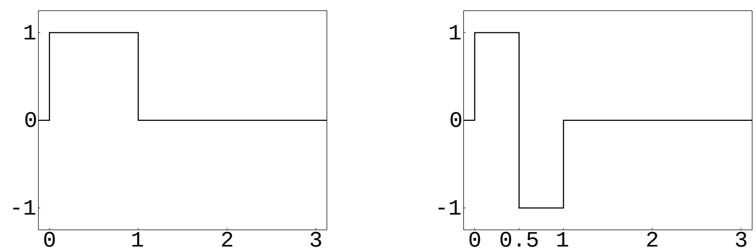
We see from (1.7) and (1.2) that the scaled vectors $U\psi_i$ and $U\varphi$ are both finite linear combinations of translates $\left\{T^k\varphi\right\}_{k\in\mathbb{Z}}$ if and only if the functions m_i are polynomials, and this is reflected in the fact that the wavelets φ, ψ_i have compact support if and only if all the functions $m_i(z)$ are polynomials in z. (See [12, Chapter 5], [21, Section 3.3].) In [4], a detailed study was made of the representations of $\mathcal{O}_N$ defined by (1.9) in the case where $m_i(z)$ are monomials (or more precisely, monomials of the form $m_i(z) = z^{n_i}$; the more general case where $m_i(z) = \lambda_i z^{n_i}$ with $\lambda_i \in \mathbb{T} \subset \mathbb{C}$ was considered in [13]). It is clear from (1.2) and (1.7) that the other m_i-functions coming from wavelets are never monomials, but the Haar wavelet (for $N = 2$), $\varphi(x) = \chi_{[0,1]}(x)$, is close: one checks from (1.2) and (1.4) that $m_0(z) = (1+z)/\sqrt{2}$. The most general choice of m_1 is then

$$m_1(z) = z f\left(z^2\right) \overline{m_0(-z)}, \tag{1.14}$$

where f maps $\mathbb{T}$ into $\mathbb{T}$, and one conventional choice is $f = -1$, i.e.,

$$m_1(z) = (1-z)/\sqrt{2}. \tag{1.15}$$

Thus the Haar mother function is given by $\frac{1}{\sqrt{2}}\psi\left(\frac{x}{2}\right) = \frac{1}{\sqrt{2}}\left(\varphi(x) - \varphi(x-1)\right)$, i.e., the graph of ψ is that represented in Figure 2.



φ ψ

FIGURE 2. Father and mother functions for the Haar wavelet

If S_i is defined by (1.9), and one transforms the representation by $\frac{1}{\sqrt{2}}\left(\begin{smallmatrix}1 & -1\\ 1 & 1\end{smallmatrix}\right) \in \mathrm{U}(2)$, i.e.,

$$T_0 = (S_0 + S_1)/\sqrt{2}, \qquad T_1 = (S_0 - S_1)/\sqrt{2}, \tag{1.16}$$

one verifies that the pair T_0, T_1 still satisfies the Cuntz relations, and

$$T_0\xi(z) = \xi\left(z^2\right), \qquad T_1\xi(z) = z\xi\left(z^2\right). \tag{1.17}$$

This is one of the monomial representations studied in [4], and by [4, Proposition 8.1], this representation of $\mathcal{O}_2$ decomposes into two inequivalent irreducible subrepresentations on the subspaces

$$H^2(\mathbb{T}) = \overline{\operatorname{span}}^{\|\cdot\|_2}\{z^n \mid n \in \mathbb{N}\cup\{0\}\}, \tag{1.18}$$

$$H^2(\mathbb{T})^\perp = \overline{zH^2(\mathbb{T})} = \overline{\operatorname{span}}^{\|\cdot\|_2}\left\{z^{-n} \mid n \in \mathbb{N}\right\}, \tag{1.19}$$

where $\overline{zH^2(\mathbb{T})}$ refers to complex conjugation, and $\overline{}^{\|\cdot\|_2}$ is $L^2(\mathbb{T})$ closure. Thus the original Haar wavelet representation is a direct sum of two inequivalent irreducible subrepresentations. In general, when the functions m_i are polynomials, this simple trick of reducing to monomials is not going to work, but we will see that it is possible to develop a theory for polynomial representations which nonetheless has many general analogues with the monomial theory.

2. Finitely correlated states on the Cuntz algebra $\mathcal{O}_N$

Let us recall a few facts about the Cuntz algebra $\mathcal{O}_N$ from [10], and the part of the results from [6] that will be needed in the sequel.

If $N \in \{2, 3, \dots\}$, the *Cuntz algebra* $\mathcal{O}_N$ is the universal C^*-algebra generated by elements $s_0, \dots, s_{N-1}$ subject to the relations

$$s_i^* s_j = \delta_{ij}\mathbb{1}, \qquad \sum_{j\in\mathbb{Z}_N} s_j s_j^* = \mathbb{1}. \tag{2.1}$$

The Cuntz algebra may be viewed as an interpolation between the algebra of the canonical anti-commutation relations (CAR) and the algebra of the canonical commutation relations (CCR): The q-canonical commutation relations,

$$a_i a_j^* - q a_j^* a_i = \delta_{ij}\mathbb{1},$$

$i, j = 1, \dots, d$, reduce to the CCR relations if $q = 1$, the CAR relations if $q = -1$, and the Cuntz relations (2.1) if $q = 0$. See [22, 7, 8, 16, 17] for details on this.

The Cuntz algebra is a simple separable C^*-algebra not isomorphic to the algebra of compact operators on a Hilbert space. Therefore the space of unitary equivalence

classes of irreducible representations of $\mathcal{O}_N$ cannot be parametrized in a measurable way [15]. In this paper we will show that the representations coming from low-pass filters of genus 2 form a (necessarily tiny) one-dimensional variety in this enormous space.

There is a canonical action of the group $U(N)$ of unitary $N \times N$ matrices on $\mathcal{O}_N$ given by

$$\tau_g(s_i) = \sum_{j \in \mathbb{Z}_N} \overline{g_{ji}} s_j \tag{2.2}$$

for $g = [g_{ij}] \in \mathrm{U}(N)$. In particular the *gauge action* is defined by $\tau_z(s_i) = z s_i$, $z \in \mathbb{T} \subset \mathbb{C}$. If UHF_N is the fixed point subalgebra under the gauge action, then UHF_N is the closure of the linear span of all Wick ordered monomials of the form $s_{i_1} \cdots s_{i_k} s_{j_k}^* \cdots s_{j_1}^*$. UHF_N is isomorphic to the UHF-algebra of Glimm type N^∞,

$$\mathrm{UHF}_N \cong M_{N^\infty} = \bigotimes_1^\infty M_N, \tag{2.3}$$

in such a way that the isomorphism carries the aforementioned Wick ordered monomial, $s_{i_1} \cdots s_{i_k} s_{j_k}^* \cdots s_{j_1}^*$, into the matrix element

$$e_{i_1 j_1} \otimes e_{i_2 j_2} \otimes \cdots \otimes e_{i_k j_k} \otimes \mathbb{1} \otimes \mathbb{1} \otimes \cdots . \tag{2.4}$$

The restriction of τ_g to UHF_N is then carried into the action

$$\mathrm{Ad}(g) \otimes \mathrm{Ad}(g) \otimes \cdots \tag{2.5}$$

on $\bigotimes_1^\infty M_N$. We define the canonical endomorphism λ on UHF_N (or on $\mathcal{O}_N$) by

$$\lambda(x) = \sum_{j \in \mathbb{Z}_N} s_j x s_j^* \tag{2.6}$$

and the isomorphism carries λ over into the one-sided shift

$$x_1 \otimes x_2 \otimes x_3 \otimes \cdots \longrightarrow \mathbb{1} \otimes x_1 \otimes x_2 \otimes \cdots \tag{2.7}$$

on $\bigotimes_1^\infty M_N$.

If $s_i \mapsto S_i \in \mathcal{B}(\mathcal{H})$ is a representation of the Cuntz relations on a Hilbert space $\mathcal{H}$, we will say (by abuse of terminology) that the representation is finitely correlated if there exists a finite-dimensional subspace $\mathcal{K} \subset \mathcal{H}$ with the two properties

(2.8) $S_i^* \mathcal{K} \subset \mathcal{K}$ for $i \in \mathbb{Z}_N$,

(2.9) $\mathcal{K}$ is cyclic for the representation $s_i \longmapsto S_i$.

The presence of such a finite-dimensional subspace $\mathcal{K}$ is a special property of each of the representations under discussion, and therefore of the states of $\mathcal{O}_N$ which correspond to the representations. These states were studied in [6] with a view to the present applications.

If $P \colon \mathcal{H} \to \mathcal{K}$ is the orthogonal projection onto $\mathcal{K}$, then (2.8) can be formulated as

$$V_i \equiv P S_i = P S_i P. \tag{2.10}$$

If we view V_i as operators in $\mathcal{B}(\mathcal{K})$, we have

$$\sum_{i \in \mathbb{Z}_N} V_i V_i^* = \mathbb{1}, \tag{2.11}$$

and conversely, if V_i are operators in $\mathcal{B}(\mathcal{K})$ satisfying (2.11), they determine a representation $s_i \mapsto S_i$ of the Cuntz relations such that (2.10) is valid, and this representation is unique up to unitary equivalence if we require $\mathcal{K}$ to be cyclic [6, Theorem 5.1].

If $\mathcal{K}_1$ is another Hilbert space and $W_0, \dots, W_{N-1}$ are operators on $\mathcal{K}_1$ satisfying

$$\sum_{i \in \mathbb{Z}_N} W_i W_i^* = \mathbb{1}, \tag{2.12}$$

and $s_i \mapsto T_i$ is the associated representation of $\mathcal{O}_N$, then there is an isometric linear isomorphism between intertwiners $U \colon \mathcal{H}_V \to \mathcal{H}_W$, i.e., operators satisfying

$$US_i = T_i U, \tag{2.13}$$

and operators $V \in \mathcal{B}(\mathcal{K}, \mathcal{K}_1)$ such that

$$\boldsymbol{\rho}(V) \equiv \sum_{i \in \mathbb{Z}_N} W_i V V_i^* = V. \tag{2.14}$$

This linear isomorphism is given by

$$U \longmapsto V = P_1 V P, \tag{2.15}$$

where $P_1 \colon \mathcal{H}_W \to \mathcal{K}_1$ is the orthogonal projection onto $\mathcal{K}_1$. All these results do not depend on $\mathcal{K}$ and $\mathcal{K}_1$ being finite-dimensional, and they are given in [6, Theorem 5.1].

An important special case is $\mathcal{K}_1 = \mathcal{K}$ and $W_i = V_i$. Then $\boldsymbol{\rho}$ is a completely positive unital map, and the linear isomorphism (2.15) is an order isomorphism between the fixed point set of $\boldsymbol{\rho}$ (which is not necessarily an algebra) and the commutant $\{S_i, S_i^* \mid i \in \mathbb{Z}_N\}'$. In particular, we have the following principle.

(2.16) The representation $s_i \longmapsto S_i$ is irreducible if and only if $\boldsymbol{\rho}$ is ergodic: $\{A \in \mathcal{B}(\mathcal{K}) \mid \boldsymbol{\rho}(A) = A\} = \mathbb{C}\mathbb{1}$.

The rest of the discussion in this section can only be partially extended to the case when $\mathcal{K}$ is infinite-dimensional (see [6, Section 6] for details). Define $\boldsymbol{\sigma} = \boldsymbol{\rho}$ in the case when $\mathcal{K}_1 = \mathcal{K}$ and $W_i = V_i$ in (2.14). If $\boldsymbol{\sigma}$ is ergodic, then $\mathcal{B}(\mathcal{K})$ has a unique $\boldsymbol{\sigma}$-invariant state φ. This state need not be faithful (see the example after the proof of Lemma 3.4 in [6]). If E is the support projection of φ, then $S_i^* E\mathcal{K} \subset E\mathcal{K}$ for all $i \in \mathbb{Z}_N$ (see [6, Lemma 6.1]). In that case, replace P by E, V_i by EV_i, $\boldsymbol{\sigma}$ by the $\boldsymbol{\sigma}$ defined by the new V_i's on $E\mathcal{K}$, and then define a state ψ on $\mathcal{O}_N$ by

$$\psi(S_I S_J^*) = \varphi(E S_I S_J^* E). \tag{2.17}$$

It was proved in [6, Theorem 6.3] that the following three subsets of the circle group $\mathbb{T}$ are equal:

(2.18) $\{t \in \mathbb{T} \mid \psi \circ \tau_t = \psi\}$, where τ is the gauge action;

(2.19) $\{t \in \mathbb{T} \mid \psi \circ \tau_t$ is quasi-equivalent to $\psi\}$;

(2.20) $\operatorname{PSp}(\boldsymbol{\sigma}) \cap \mathbb{T}$, where $\operatorname{PSp}(\boldsymbol{\sigma})$ is the set of eigenvalues of $\boldsymbol{\sigma}$.

(Of course, in the present setting, where $E\mathcal{K}$ is finite-dimensional, $\operatorname{PSp}(\boldsymbol{\sigma}) = \operatorname{Sp}(\boldsymbol{\sigma})$.) Furthermore, this subset is a finite subgroup of $\mathbb{T}$. If k is the order of this subgroup, the restriction of the representation to UHF_N decomposes into k mutually disjoint irreducible representations, and these are mapped cyclically one into another by the one-sided shift λ. More specifically, one has $\operatorname{PSp}(\boldsymbol{\sigma}) \cap \mathbb{T} = \operatorname{PSp}(\lambda) \cap \mathbb{T}$, and, if $t_k = e^{\frac{2\pi i}{k}}$, there exists a unitary U on $\mathcal{H}$, unique up to a scalar,

implementing τ_{t_k}, and such that $U^k = \mathbb{1}$. The operator U is the unique (up to a scalar) eigen-element such that $\lambda(U) = \bar{t}_k U$. If

$$U = \sum_{n \in \mathbb{Z}_k} t_k^n E_n \tag{2.21}$$

is the spectral decomposition of U, then the spectral projections E_n project into mutually disjoint irreducible subspaces invariant for the representation restricted to UHF_N, and $\lambda(E_n) = E_{n+1}$, with λ extended to $\mathcal{B}(\mathcal{H})$ by the formula $\lambda(\,\cdot\,) = \sum_{i \in \mathbb{Z}_N} S_i \cdot S_i^*$.

3. Polynomial representations

From the relation (1.9) it follows that

$$(S_i^* \xi)(z) = \frac{1}{N} \sum_{w^N = z} \overline{m_i(w)} \xi(w), \tag{3.1}$$

where the sum ranges over all N'th roots w of z [5, eq. (1.17)]. Recall that the Fourier series version of (3.1) on $\ell^2(\mathbb{Z})$ is the filter operator F_i of (1.12). In order to incorporate the monomial results obtained in [4], and also to make the present results applicable to wavelets in dimension $\nu > 1$, let us extend the definitions of the representations somewhat. We replace $L^2(\mathbb{T})$ with $L^2(\mathbb{T}^\nu)$ and fix a matrix $\mathbf{N}$ with integer coefficients such that $|\det(\mathbf{N})| = N \in \{2, 3, \dots\}$. If $z = (z_1, \dots, z_\nu) \in \mathbb{T}^\nu$ define

$$z^{\mathbf{N}} = (z_1^{n_{11}} \cdots z_\nu^{n_{\nu 1}}, \dots, z_1^{n_{1\nu}} \cdots z_\nu^{n_{\nu\nu}}) \in \mathbb{T}^\nu \tag{3.2}$$

if $\mathbf{N} = [n_{ij}]_{i,j=1}^{\nu}$. (Note that this definition of $z^{\mathbf{N}}$ is different from the one after (1.8) in [4]. The present convention implies that relations like $\left(z^{\mathbf{N}}\right)^{\mathbf{M}} = z^{\mathbf{NM}}$ and $\left(z^{\mathbf{N}}\right)^n = z^{\mathbf{N}n}$ are valid, where z^n is defined as in connection with (3.6) below. The present map $z \mapsto z^{\mathbf{N}}$ is the transpose of the map $x \mapsto \mathbf{N}x$ on $\mathbb{R}^\nu$ passed to the quotient $\mathbb{T}^\nu = \mathbb{R}^\nu / 2\pi\mathbb{Z}^\nu$.) The map $z \mapsto z^{\mathbf{N}}$ is N-to-1. Let $\sigma_0, \dots, \sigma_{N-1}$ denote sections of this map, i.e., each $\sigma_i \colon \mathbb{T}^\nu \to \mathbb{T}^\nu$ is injective, $\mu(\sigma_i(\mathbb{T}^\nu) \cap \sigma_j(\mathbb{T}^\nu)) = 0$ if $i \neq j$, where μ is normalized Haar measure on $\mathbb{T}^\nu$, and $\mu(\sigma_i(Y)) = \frac{1}{N}\mu(Y)$ for all Borel sets $Y \subset \mathbb{T}^\nu$. Thus $\bigcup_{i \in \mathbb{Z}_N} \sigma_i(\mathbb{T}^\nu) = \mathbb{T}^\nu$ up to sets of measure zero. The unitarity condition (1.6) then says that the $N \times N$ matrix

$$\frac{1}{\sqrt{N}} \begin{pmatrix} m_0(\sigma_0(z)) & m_0(\sigma_1(z)) & \dots & m_0(\sigma_{N-1}(z)) \\ m_1(\sigma_0(z)) & m_1(\sigma_1(z)) & \dots & m_1(\sigma_{N-1}(z)) \\ \vdots & \vdots & \ddots & \vdots \\ m_{N-1}(\sigma_0(z)) & m_{N-1}(\sigma_1(z)) & \dots & m_{N-1}(\sigma_{N-1}(z)) \end{pmatrix} \tag{3.3}$$

is unitary for almost all $z \in \mathbb{T}^\nu$. The representation (1.9), (3.1) of $\mathcal{O}_N$ now takes the form

$$(S_i \xi)(z) = m_i(z)\, \xi\left(z^{\mathbf{N}}\right), \tag{3.4}$$

and then

$$(S_i^* \xi)(z) = \frac{1}{N} \sum_{w^{\mathbf{N}} = z} \overline{m_i(w)} \xi(w). \tag{3.5}$$

Now, assume in addition to unitarity of (3.3) that $m_0, \dots, m_{N-1}$ all are polynomials, so that there exists a fixed finite subset $D \subset \mathbb{Z}^\nu$ such that

$$m_j(z) = \sum_{n \in D} a_n^{(j)} z^n. \tag{3.6}$$

Here we have used the notation $z^n = (z_1, \dots, z_\nu)^{(n_1, \dots, n_\nu)} = z_1^{n_1} z_2^{n_2} \cdots z_\nu^{n_\nu}$, and $a_n^{(j)} \in \mathbb{C}$. Let e_n, $n \in \mathbb{Z}^\nu$, denote the usual Fourier basis for $L^2(\mathbb{Z}^\nu)$, i.e., $e_n(z) = z^n$. It follows from (3.4) that

$$S_j e_n = \sum_{k \in D} a_k^{(j)} e_{k + \mathbf{N} n}. \tag{3.7}$$

If in general we define $a_k^{(j)} = 0$ when $k \notin D$, it follows from (3.7) or (3.5) that

$$S_j^* e_n = \sum_{m \in \mathbb{Z}^\nu} \overline{a_{n - \mathbf{N} m}^{(j)}} e_m = \sum_{p \in D\colon\, p = n \bmod \mathbf{N}} \overline{a_p^{(j)}} e_{\mathbf{N}^{-1}(n-p)}. \tag{3.8}$$

Thus both S_j and S_j^* map trigonometric polynomials into trigonometric polynomials in this case. If the matrix $\mathbf{N}^{-1}$ defines a contractive map $\mathbb{R}^\nu \to \mathbb{R}^\nu$ in some norm, one can say more. The following proposition is an analogue of Lemma 3.8 in [4] in the present setting.

Proposition 3.1. *Assume that all the* (*complex*) *eigenvalues of* $\mathbf{N}$ *have modulus greater than* 1*. It follows that there is a finite subset* $H \subset \mathbb{Z}^\nu$ *with the property that for any* $n \in \mathbb{Z}^\nu$ *there exists an* $M \in \mathbb{N}$ *such that*

$$S_I^* e_n \in \widehat{\ell^2(H)} \equiv \operatorname{span}\{e_m \mid m \in H\} \tag{3.9}$$

for all multi-indices I *with* $|I| \geq M$*.*

Proof. Let us give two proofs of this statement, both based on a study of the maps $\sigma_p \colon \mathbb{R}^\nu \to \mathbb{R}^\nu$ defined for $p \in D$ by

$$\sigma_p(x) = \mathbf{N}^{-1}(x - p) \tag{3.10}$$

for $x \in \mathbb{R}^\nu$. By considering a Jordan form of $\mathbf{N}$, as in the proof of Lemma 3.8 in [4], the condition $|\lambda_i| > 1$ on the eigenvalues of $\mathbf{N}$ means that there exists a norm on $\mathbb{C}^\nu$ such that $\left\|\mathbf{N}^{-1}\right\| < 1$ in the associated norm on $\mathcal{B}(\mathbb{C}^\nu)$. If $d = \max\{\|p\| \mid p \in D\}$, it follows from (3.10) that $\|\sigma_p(x)\| \leq \left\|\mathbf{N}^{-1}\right\|(\|x\| + d)$ for $p \in D$, and by iteration,

$$\begin{aligned} \|\sigma_{p_1}\sigma_{p_2}\cdots\sigma_{p_n}(x)\| &\leq \left\|\mathbf{N}^{-1}\right\|^n \|x\| + \sum_{k=1}^n \left\|\mathbf{N}^{-1}\right\|^k d \\ &= \left\|\mathbf{N}^{-1}\right\|^n \|x\| + \left\|\mathbf{N}^{-1}\right\| \frac{1 - \left\|\mathbf{N}^{-1}\right\|^n}{1 - \left\|\mathbf{N}^{-1}\right\|} d \leq \left\|\mathbf{N}^{-1}\right\|^n \|x\| + \frac{\left\|\mathbf{N}^{-1}\right\|}{1 - \left\|\mathbf{N}^{-1}\right\|} d \end{aligned} \tag{3.11}$$

for $p_1, \dots, p_n \in D$, $n \in \mathbb{N}$. Now, using (3.8) in the form

$$S_j^* e_n = \sum_{p \in D\colon\, p = n \bmod \mathbf{N}} \overline{a_p^{(j)}} e_{\sigma_p(n)}, \tag{3.12}$$

one deduces from (3.11) that

$$\begin{aligned} &S_I^* \widehat{\ell^2}(\{m \in \mathbb{Z}^\nu \mid \|m\| \leq R\}) \\ &\quad\subset \widehat{\ell^2}\left(\left\{m \in \mathbb{Z}^\nu \;\middle|\; \|m\| \leq \left\|\mathbf{N}^{-1}\right\|^{|I|} R + \left(\left\|\mathbf{N}^{-1}\right\| / \left(1 - \left\|\mathbf{N}^{-1}\right\|\right)\right) d\right\}\right). \end{aligned} \tag{3.13}$$

Thus Proposition 3.1 follows with

$$H = \left\{ n \in \mathbb{Z}^\nu \;\middle|\; \|n\| \leq \left(\left\| \mathbf{N}^{-1} \right\| / \left(1 - \left\| \mathbf{N}^{-1} \right\| \right) \right) d \right\}. \tag{3.14}$$

□

Remark 3.2. The other method of proving Proposition 3.1 is a small variation which gives an optimal choice of H given only D. By a theorem of Bandt [1, 2, 11, 29] cited in [4, (3.11)–(3.12)] there is a unique compact subset $X \subset \mathbb{R}^\nu$ such that X is a fixed point for the map $Y \mapsto \bigcup_{p \in D} \sigma_p(Y)$, i.e.,

$$X = \bigcup_{p \in D} \sigma_p(X), \tag{3.15}$$

and we may take

$$H = X \cap \mathbb{Z}^\nu = H(D). \tag{3.16}$$

In some examples in Section 4, the finite subset $H \subset \mathbb{Z}^\nu$ will be computed explicitly. If the representation of $\mathcal{O}_N$ is irreducible, an application of [6, Lemma 6.1] further shows that the finite-dimensional subspace $\mathcal{K}(H)$ from (3.16) contains a unique minimal subspace $\mathcal{M} \neq 0$ with the invariance property $S_i^* \mathcal{M} \subset \mathcal{M}$.

The following corollary is the main tool in analyzing polynomial representation.

Corollary 3.3. *Consider the polynomial representation of $\mathcal{O}_N$ defined by* (3.7) *and* (3.8)*, and let H be a minimal finite subset of $\mathbb{Z}^\nu$ satisfying the properties in Proposition* 3.1*. It follows that*

$$\mathcal{K} = \widehat{\ell^2(H)} \tag{3.17}$$

is cyclic for the representation, and thus the representation is finitely correlated. Defining $V_j^* \in \mathcal{B}(\mathcal{K})$ by

$$V_j^* e_n = \sum_{m \in H} \overline{a_{n - \mathbf{N}m}^{(j)}} e_m = \sum_{\substack{p \in D:\; p = n \bmod \mathbf{N} \\ \sigma_p(n) \in H}} \overline{a_p^{(j)}} e_{\sigma_p(n)} \tag{3.18}$$

for $n \in H$, the commutant of the representation is isometrically order isomorphic to

$$\mathcal{B}(\mathcal{K})^{\boldsymbol{\sigma}} = \left\{ A \in \mathcal{B}(\mathcal{K}) \;\middle|\; \boldsymbol{\sigma}(A) \equiv \sum_{k \in \mathbb{Z}_N} V_k A V_k^* = A \right\}. \tag{3.19}$$

In particular the representation is irreducible if and only if $\mathcal{B}(\mathcal{K})^{\boldsymbol{\sigma}} = \mathbb{C}\mathbb{1}$. In this case, the peripheral spectrum of $\boldsymbol{\sigma}$ is always a finite (necessarily cyclic) subgroup of $\mathbb{T}$, and if k is the order of this subgroup, the restriction of the representation to UHF_N *decomposes into the direct sum of k mutually disjoint irreducible representations.*

In general the intertwiner space between two representations of this type is given by (2.13)–(2.14)*.*

Proof. The identity

$$\mathbb{1} = \sum_{I:\, |I| = M} S_I S_I^*, \tag{3.20}$$

in conjunction with Proposition 3.1, implies that all monomials e_n, $n \in \mathbb{Z}^\nu$, are contained in the cyclic subspace generated by $\mathcal{K}$, and hence this space is dense in

$L^2(\mathbb{T}^\nu)$. Indeed, for every $n \in \mathbb{Z}^\nu$, there is, by Proposition 3.1, an $M \in \mathbb{N}$ such that $S_I^* e_n \in \mathcal{K}$ for all I such that $|I| \geq M$. Therefore $S_I S_I^* e_n \in S_I(\mathcal{K})$. An application of (3.20) to e_n then yields the desired cyclicity. This cyclicity is the second of the two properties of the subspace $\mathcal{K}$ in the discussion of Section 2, i.e., (2.9). The rest (and some more details) follows from the discussion in Section 2. □

4. Classification of some polynomial representations

If D is a given finite subset of $\mathbb{Z}^\nu$, the set of all polynomials m_j given by (3.6), and satisfying the unitarity condition (3.3) and the normalization

$$m_0(1) = \sqrt{N} \tag{4.1}$$

(which is necessary for the convergence of the Mallat expansion; see [23] or [5, eq. (1.37)]), forms a compact algebraic variety $\mathcal{M}_D$, and it is given as the solution variety of a set of quadratic equations in the coefficients $a_n^{(j)}$ and $\overline{a_n^{(j)}}$ with $n \in D$. For each point on this variety $\mathcal{M}_D$, the corresponding representation of $\mathcal{O}_N$ can in principle be computed from Corollary 3.3. Even the characterization of $\mathcal{M}_D$ is a formidable task in general, but it has been done in the case $\nu = 1$ and $N = 2$ in [30, 28, 24, 20, 19] and [25] (see also [26, 27]). In this section, we will compute the representation theory of $\mathcal{O}_2$ for each of the points of some of these varieties. We do not know if our results indicate how the generic behaviour of this representation theory will be, but in the examples the representations generically are irreducible and mutually disjoint, with exceptional behaviour on a sub-variety of lower dimension.

4.1. The case with dimension $\nu = 1$.

In this case, $\mathbf{N} = N \in \{2, 3, 4, \dots\}$. If $m_j(z) = \sum_{n \in D} a_n^{(j)} z^n$, where $m = m_i$ for some i, unitarity of (1.6) implies

$$\sum_{k \in \mathbb{Z}_N} \left| m_j\left(\rho^k z\right) \right|^2 = N, \tag{4.2}$$

which is equivalent to the conditions

$$\sum_n a_n^{(j)} \overline{a_n^{(j)}} = 1 \quad \text{and} \quad \sum_n a_n^{(j)} \overline{a_{n-mN}^{(j)}} = 0 \tag{4.3}$$

for $m = 1, 2, \dots$. Analogously, orthogonality of the rows in (1.6) leads to

$$\sum_n a_n^{(i)} \overline{a_{n-mN}^{(j)}} = 0 \tag{4.4}$$

for all $i \neq j$ and *all* $m \in \mathbb{Z}$. Finally, the normalization (4.1) leads to

$$\sum_n a_n^{(0)} = \sqrt{N}. \tag{4.5}$$

The relations (4.3)–(4.5), together with $a_n^{(i)} = 0$ for $n \notin D$, determine the algebraic variety $\mathcal{M}_D$. Let us now restrict to $N = 2$, and to the case where the $a_n^{(j)}$'s are real (this latter assumption, reality, seems conventional in wavelet theory). Then by (1.14),

$$m_1(z) = z f\left(z^2\right) \overline{m_0(-z)}, \tag{4.6}$$

where f is a monomial. By translating the father, and mother, functions by multiples of T (integral translations), we may assume that D has the form

$$D = \{0, 1, \dots, 2d-1\}, \tag{4.7}$$

where $d \in \mathbb{N}$, and with $f(w) = -w^{d-1}$, we have

$$m_0(z) = \sum_{k=0}^{2d-1} a_k z^k, \tag{4.8}$$

$$m_1(z) = \sum_{k=0}^{2d-1} (-1)^{k+1} a_k z^{2d-1-k} = \sum_{k=0}^{2d-1} (-1)^k a_{2d-1-k} z^k. \tag{4.9}$$

The conditions (4.3)–(4.5) then become

$$\sum_{k=0}^{2d-1} a_k^2 = 1, \quad \text{and} \quad \sum_{k=0}^{2(d-m)-1} a_k a_{k+2m} = 0, \tag{4.10}$$

for $m = 1, \dots, d-1$ (no condition if $d = 1$), and

$$\sum_{k=0}^{2d-1} a_k = \sqrt{2}. \tag{4.11}$$

(The condition (4.4) is already taken care of in (4.9).)

In this case the maps σ_p in (3.10) have the form

$$\sigma_p(x) = \frac{x-p}{2} \tag{4.12}$$

for $p = 0, 1, \dots, 2d-1$, and thus the solution X to the equation (3.15) is the interval

$$X = [-2d+1, 0], \tag{4.13}$$

and hence by (3.16)

$$H = \{-2d+1, -2d+2, \dots, 0\}, \tag{4.14}$$

$$\mathcal{K} = \operatorname{span}\{e_{-2d+1}, e_{-2d+2}, \dots, e_0\}. \tag{4.15}$$

It follows from (3.18) that the matrix for V_0^* relative to the basis $\{e_0, e_{-1}, \dots, e_{-2d+1}\}$ has the form (passing under the name "slant-Toeplitz matrix")

(4.16)

$$\begin{pmatrix}
a_0 & 0 & 0 & 0 & 0 & \cdots\cdots & 0 & 0 & 0 & 0 & 0 \\
a_2 & a_1 & a_0 & 0 & 0 & \cdots\cdots & 0 & 0 & 0 & 0 & 0 \\
a_4 & a_3 & a_2 & a_1 & a_0 & \cdots\cdots & 0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots \\
a_{2d-4} & a_{2d-5} & a_{2d-6} & a_{2d-7} & a_{2d-8} & \cdots\cdots & a_1 & a_0 & 0 & 0 & 0 \\
a_{2d-2} & a_{2d-3} & a_{2d-4} & a_{2d-5} & a_{2d-6} & \cdots\cdots & a_3 & a_2 & a_1 & a_0 & 0 \\
0 & a_{2d-1} & a_{2d-2} & a_{2d-3} & a_{2d-4} & \cdots\cdots & a_5 & a_4 & a_3 & a_2 & a_1 \\
0 & 0 & 0 & a_{2d-1} & a_{2d-2} & \cdots\cdots & a_7 & a_6 & a_5 & a_4 & a_3 \\
\vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & \cdots\cdots & a_{2d-1} & a_{2d-2} & a_{2d-3} & a_{2d-4} & a_{2d-5} \\
0 & 0 & 0 & 0 & 0 & \cdots\cdots & 0 & 0 & a_{2d-1} & a_{2d-2} & a_{2d-3} \\
0 & 0 & 0 & 0 & 0 & \cdots\cdots & 0 & 0 & 0 & 0 & a_{2d-1}
\end{pmatrix}$$

and the matrix for V_1^* is, by (4.9), obtained by using the substitution $a_k \to (-1)^k a_{2d-1-k}$ in the matrix (4.16). Note that the subspace

$$\mathcal{K}_0 = \operatorname{span}\{e_{-2d+2}, e_{-2d+3}, \dots, e_{-1}\} \tag{4.17}$$

is also invariant under V_0^* and V_1^*, and thus under S_0^* and S_1^*, but we will see in Section 4.1.2.2 below that this subspace is not always cyclic.

Let us remark that the scaling relations for the father function φ corresponding to (4.8) and the mother function ψ from (4.9) (both in $L^2(\mathbb{R})$) are as follows:

$$\frac{1}{\sqrt{2}}\varphi\left(\frac{x}{2}\right) = \sum_k a_k \varphi(x-k), \tag{4.18}$$

$$\frac{1}{\sqrt{2}}\psi\left(\frac{x}{2}\right) = \sum_k (-1)^k a_{2d-1-k}\varphi(x-k). \tag{4.19}$$

See also Remark 4.3.

Following the terminology in [30], we say that d is the *genus*, and we now turn to a closer study of $d \le 2$.

4.1.1. *The case with dimension $\nu = 1$, scale $N = 2$, and genus $d = 1$.* In this case, the second condition of (4.10) is vacuous, and the only solution of (4.10) and (4.11) is $a_0 = a_1 = \frac{1}{\sqrt{2}}$, so

$$m_0(z) = (1+z)/\sqrt{2}, \qquad m_1(z) = (1-z)/\sqrt{2}, \tag{4.20}$$

which is exactly the Haar wavelet (Figure 2). The representation splits into the direct sum of the two inequivalent irreducible representations in (1.18) and (1.19), and the restriction of each of these representations to UHF_2 is still irreducible by [4, Proposition 8.1]. This can also be checked directly: in this case,

$$V_0^* = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix}, \qquad V_1^* = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & -\frac{1}{\sqrt{2}} \end{pmatrix}. \tag{4.21}$$

Thus

$$\boldsymbol{\sigma}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \sum_{i=0}^{1} V_i \begin{pmatrix} a & b \\ c & d \end{pmatrix} V_i^* = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}, \tag{4.22}$$

so $\mathcal{B}(\mathcal{K})^{\boldsymbol{\sigma}}$ is the $*$-algebra of all diagonal 2×2 matrices. Thus the representation splits into the direct sum of two representations with the one-dimensional S_i^*-invariant subspaces $\mathbb{C}e_0$ and $\mathbb{C}e_{-1}$. The corresponding maps $\boldsymbol{\sigma}$ on the one-dimensional subspaces are both equal to the identity, thus they are ergodic with peripheral spectrum 1, and UHF_2 is dense by Corollary 3.3. Note that the states on $\mathcal{O}_2$ corresponding to e_0 and e_{-1} are the Cuntz states (see [10, 16, 6])

$$\omega_0(S_I S_J^*) = 2^{-\frac{|I|+|J|}{2}}, \qquad \omega_{-1}(S_I S_J^*) = (-1)^{|I|+|J|} 2^{-\frac{|I|+|J|}{2}}. \tag{4.23}$$

4.1.2. *The case with dimension $\nu = 1$, scale $N = 2$, and genus $d = 2$.* We now display the one-parameter family (with two singular points) of mutually inequivalent irreducible representations of $\mathcal{O}_2$ mentioned in the Introduction, and we relate the representation-theoretic behavior to the corresponding properties of the associated

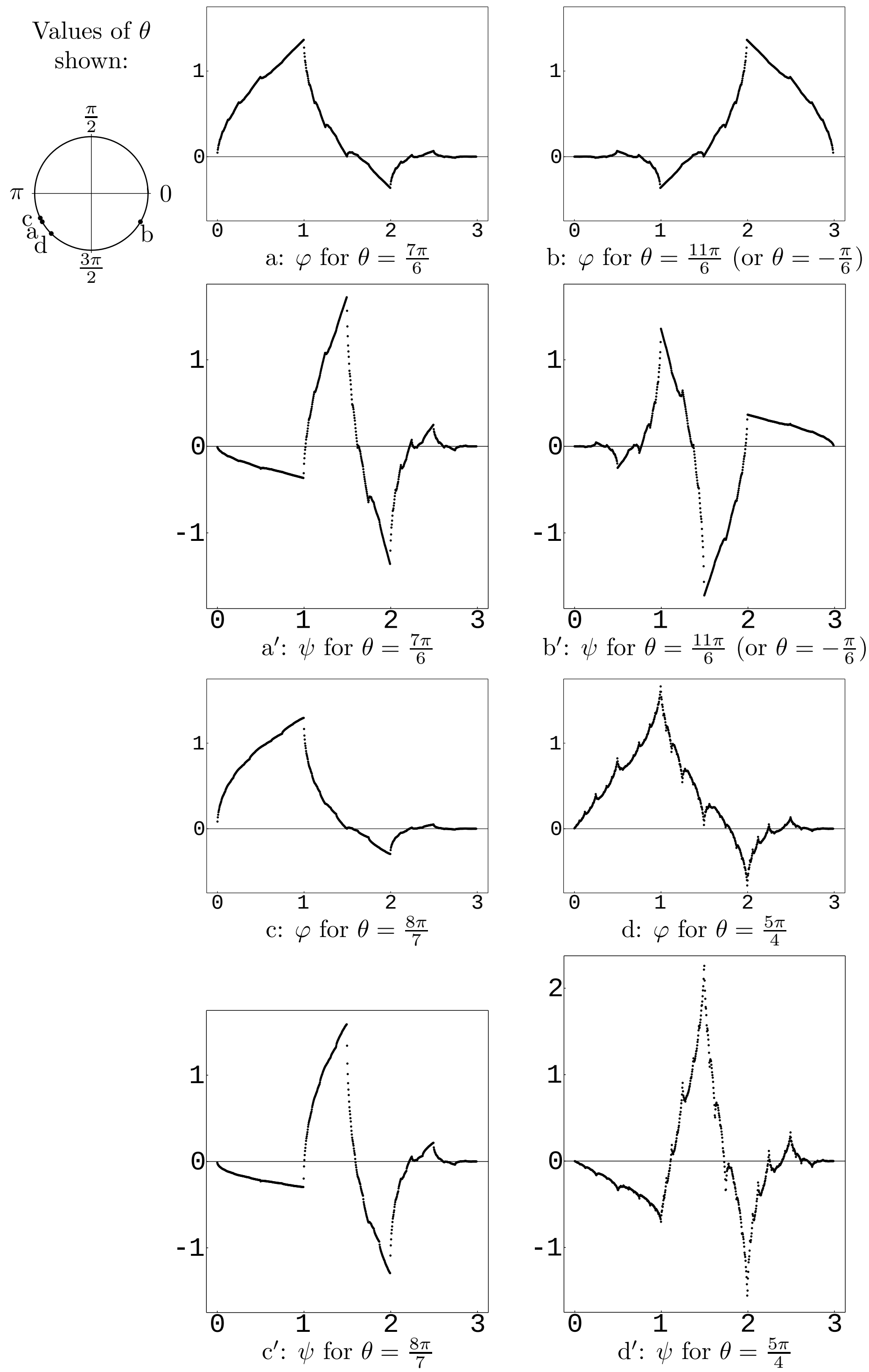


FIGURE 3. Father (φ) and mother (ψ) functions for θ near $\frac{7\pi}{6}$ and $-\frac{\pi}{6}$: Continuous cases (Case "a" = Daubechies wavelet)

family of wavelets on $L^2(\mathbb{R})$. In this case the algebraic variety defined by (4.10)–(4.11) is actually the circle, and may be defined by the following parametrization:

$$\begin{aligned} a_0 &= \frac{1}{2\sqrt{2}}(1-\cos\theta+\sin\theta), & a_1 &= \frac{1}{2\sqrt{2}}(1-\cos\theta-\sin\theta), \\ a_2 &= \frac{1}{2\sqrt{2}}(1+\cos\theta-\sin\theta), & a_3 &= \frac{1}{2\sqrt{2}}(1+\cos\theta+\sin\theta); \end{aligned} \tag{4.24}$$

see [25, 26], and also [30, 19]. Let us give a simple argument for this parametrization: View $a = (a_0, a_1, a_2, a_3)$ as a function on the cyclic group of order 4, $\mathbb{Z}_4$, and consider the Fourier transform on $\mathbb{Z}_4$:

$$\hat{a}(n) = \frac{1}{2}\sum_{m=0}^{3} i^{nm} a(m), \qquad a(m) = \frac{1}{2}\sum_{n=0}^{3} i^{-nm}\hat{a}(n).$$

We have the usual formulae

$$\sum_m \overline{a(m)}b(m) = \sum_n \overline{\hat{a}(n)}\hat{b}(n), \qquad a\widehat{(\,\cdot\,+k)}(n) = i^{-nk}\hat{a}(n),$$

and thus

$$\sum_m \overline{a(m+2)}a(m) = \sum_n (-1)^n\,\overline{\hat{a}(n)}\hat{a}(n).$$

Also $\hat{\bar{a}}(n) = \overline{\hat{a}(-n)}$. The relations (4.3) and (4.5), together with reality of a, take the form

$$\sum_n a_n = \sqrt{2}, \qquad \sum_n a_n^2 = \sum_n \bar{a}_n a_n = 1, \qquad a_n = \bar{a}_n,$$
$$a_0a_2 + a_1a_3 = 0 \Longleftrightarrow \sum_m \overline{a(m+2)}a(m) = 0,$$

and hence

$$\hat{a}(0) = \frac{1}{\sqrt{2}}, \qquad \sum_n |\hat{a}(n)|^2 = 1,$$
$$\overline{\hat{a}(-n)} = a(n) \Longleftrightarrow \hat{a}(0), \hat{a}(2) \text{ are real and } \hat{a}(3) = \overline{\hat{a}(1)},$$
$$\sum_n (-1)^n\,\overline{\hat{a}(n)}\hat{a}(n) = 0.$$

Introducing $c = a(2) = \bar{c}$ and $b = \hat{a}(1)$ we thus have $c^2 + 2|b|^2 = 1/2$, $c^2 - 2|b|^2 = -1/2$, and hence $c = 0$, $|b| = 1/2$. Putting $b = \frac{1}{2}e^{i\varphi}$, the relations for a are thus equivalent to

$$(\hat{a}(0), \hat{a}(1), \hat{a}(2), \hat{a}(3)) = \left(\frac{1}{\sqrt{2}}, \frac{e^{i\varphi}}{2}, 0, \frac{e^{-i\varphi}}{2}\right).$$

Applying the inverse Fourier transform to this, we obtain

$$\begin{aligned} a_0 &= \frac{1}{2\sqrt{2}}\left(1+\sqrt{2}\cos\varphi\right), & a_1 &= \frac{1}{2\sqrt{2}}\left(1+\sqrt{2}\sin\varphi\right), \\ a_2 &= \frac{1}{2\sqrt{2}}\left(1-\sqrt{2}\cos\varphi\right), & a_3 &= \frac{1}{2\sqrt{2}}\left(1-\sqrt{2}\sin\varphi\right). \end{aligned}$$

Substituting $\varphi = \theta + \frac{5\pi}{4}$ here, we obtain (4.24).

Returning to the representation, the operators V_k^* from (4.16) in this case have the form:

$$V_0^* = \begin{pmatrix} a_0 & 0 & 0 & 0 \\ a_2 & \boxed{\begin{matrix} a_1 & a_0 \\ a_3 & a_2 \end{matrix}} & & 0 \\ 0 & & & a_1 \\ 0 & 0 & 0 & a_3 \end{pmatrix} \quad \text{and} \quad V_1^* = \begin{pmatrix} a_3 & 0 & 0 & 0 \\ a_1 & \boxed{\begin{matrix} -a_2 & a_3 \\ -a_0 & a_1 \end{matrix}} & & 0 \\ 0 & & & -a_2 \\ 0 & 0 & 0 & -a_0 \end{pmatrix}. \tag{4.25}$$

If one replaces the angle variable θ with φ, and calls the corresponding coefficients $b_0, \dots, b_3$, and the corresponding matrices W_0^*, W_1^*, the corresponding map $\boldsymbol{\rho}\colon M_4 \to M_4$ given by (2.14),

$$\boldsymbol{\rho}(A) = \sum_{i=0}^{1} W_i A V_i^*, \tag{4.26}$$

is defined by a 16×16 matrix relative to the basis

$$e_{0,0},\ e_{0,-1},\ \dots,\ e_{0,-3},\ e_{-1,0},\ e_{-1,-1},\ \dots,\ e_{-3,-3} \tag{4.27}$$

of M_4. This 16×16 matrix has the form

$$\begin{pmatrix} A_0 & A_2 & 0 & 0 \\ 0 & \boxed{\begin{matrix} A_1 & A_3 \\ A_0 & A_2 \end{matrix}} & & 0 \\ 0 & & & 0 \\ 0 & 0 & A_1 & A_3 \end{pmatrix}, \tag{4.28}$$

where the 4×4 matrices A_i are given by

$$\begin{aligned} A_0 &= b_0 V_0 + b_3 V_1, \qquad & A_1 &= b_1 V_0 - b_2 V_1, \\ A_2 &= b_2 V_0 + b_1 V_1, \qquad & A_3 &= b_3 V_0 - b_0 V_1. \end{aligned} \tag{4.29}$$

Thus one can compute the eigenvalues of $\boldsymbol{\rho}$ by computing the eigenvalues of the matrices A_0, A_3, and $\left(\begin{smallmatrix} A_1 & A_3 \\ A_0 & A_2 \end{smallmatrix}\right)$. If $\varphi = \theta$ the result is (we call $\boldsymbol{\rho} = \boldsymbol{\sigma}$ in this case as usual)

(4.30)

Eigenvalue of $\boldsymbol{\sigma}$	1	0	$\dfrac{\cos\theta}{2}$	$-\dfrac{\cos\theta}{2}$	$\dfrac{1+\sin\theta}{2}$	$-\sin\theta$
Multiplicity	1	8	2	2	2	1.

Hence, the dimension of the eigenspace $\{A \mid \boldsymbol{\sigma}(A) = A\}$ is

$$\begin{cases} 3 & \text{if } \theta = \frac{\pi}{2} \\ 2 & \text{if } \theta = \frac{3\pi}{2} \\ 1 & \text{otherwise.} \end{cases} \tag{4.31}$$

These numbers are then the dimensions of the commutants of the corresponding representations. Since the only C^*-algebras of dimensions 1, 2, 3 are $\mathbb{C}$, $\mathbb{C}^2$, $\mathbb{C}^3$, it follows that the representation of $\mathcal{O}_2$ splits into 2 inequivalent irreducible representations if $\theta = \frac{3\pi}{2}$, into 3 inequivalent irreducible representations when $\theta = \frac{\pi}{2}$, and the representation is irreducible for all other θ. We note that the peripheral spectrum of $\boldsymbol{\sigma}$ is nontrivial only if $\theta = \frac{\pi}{2}$, when -1 is an eigenvalue of multiplicity 1. Thus the representations for generic $\theta \notin \left\{\frac{\pi}{2}, \frac{3\pi}{2}\right\}$ also have irreducible restriction to UHF_2. Finally, if one considers the case $\theta \neq \varphi$, one can compute that 1 is an eigenvalue for $\boldsymbol{\rho}$ if and only if $\{\theta, \varphi\} = \left\{\frac{\pi}{2}, \frac{3\pi}{2}\right\}$, and the dimension of the corresponding eigenspace is then 2. We recall from (2.14) that solutions $A \neq 0$ to $\boldsymbol{\rho}(A) = A$ correspond by lifting to operators on $L^2(\mathbb{T})$ which intertwine the two

associated $\mathcal{O}_2$-representations $\boldsymbol{\pi}^{(\theta)}$ and $\boldsymbol{\pi}^{(\varphi)}$ for θ and φ, respectively. Hence the representations for generic points $\theta \notin \left\{\frac{\pi}{2}, \frac{3\pi}{2}\right\}$ on the circle are all mutually disjoint by (2.13)–(2.15), but if $\{\theta, \varphi\} = \left\{\frac{\pi}{2}, \frac{3\pi}{2}\right\}$, the intertwiner space is 2-dimensional. See Section 4.1.2.3 for more details on the latter.

A second immediate observation on (4.24) is that at the four points $\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$, we have two of the four coefficients vanishing with different pairs in the four different cases, so those four cases are closely connected to four modified Haar wavelets, illustrated in Figures 5 and 4. A more subtle fact, to be described below, is that it is only the two cases $\theta = \frac{\pi}{2}$ and $\theta = \frac{3\pi}{2}$ on the symmetry axis where the corresponding $\mathcal{O}_2$-representation on $L^2(\mathbb{T})$ fails to be irreducible. The case $\theta = \frac{\pi}{2}$ is degenerate in a sense illustrated in Figure 5. We will relate the resulting degenerate decomposition at $\theta = \frac{\pi}{2}$ of the subalgebra $\mathrm{UHF}_2 \subset \mathcal{O}_2$ to the wavelet properties.

Let us now consider the two exceptional points $\theta = \frac{\pi}{2}$ and $\theta = \frac{3\pi}{2}$ separately.

4.1.2.1. *The case* $\theta = \frac{3\pi}{2}$. When $\theta = \frac{3\pi}{2}$, $a_0 = a_3 = 0$, $a_1 = a_2 = 1/\sqrt{2}$, so

$$m_0(z) = \left(z + z^2\right)/\sqrt{2}, \qquad \varphi(x/2) = \varphi(x-1) + \varphi(x-2), \tag{4.32}$$

$$m_1(z) = \left(-z + z^2\right)/\sqrt{2}, \qquad \psi(x/2) = -\varphi(x-1) + \varphi(x-2), \tag{4.33}$$

with the scaling relations indicated for the father function φ, and the mother function ψ, respectively; see Figure 4.

This is a simple transform of the Haar wavelet (Figure 2), and the representation theory becomes similar: defining S_i by (1.9) and transforming the representation by $\frac{1}{\sqrt{2}}\left(\begin{smallmatrix} 1 & 1 \\ -1 & 1 \end{smallmatrix}\right) \in \mathrm{U}(2)$, i.e.,

$$T_0 = (S_0 - S_1)/\sqrt{2}, \qquad T_1 = (S_0 + S_1)/\sqrt{2}, \tag{4.34}$$

we obtain

$$T_0\xi(z) = z\xi\left(z^2\right), \qquad T_1\xi(z) = z^2\xi\left(z^2\right). \tag{4.35}$$

By the computation in [4, eqs. (8.1)–(8.2)], if U is the unitary operator given by multiplication by z^{-1}, then

$$U^*T_0U\xi(z) = \xi\left(z^2\right), \qquad U^*T_1U\xi(z) = z\xi\left(z^2\right). \tag{4.36}$$

By [4, Proposition 8.1], $L^2(\mathbb{T})$ splits into the two irreducible subspaces spanned by$\left\{1, z, z^2, \dots\right\}$ and $\left\{z^{-1}, z^{-2}, \dots\right\}$. Applying U to these, we obtain the two irreducible invariant subspaces corresponding to the original representation

$$\overline{\mathrm{span}}\left\{z^{-1}, 1, z, z^2, \dots\right\} \quad \text{and} \quad \overline{\mathrm{span}}\left\{z^{-2}, z^{-3}, \dots\right\} \tag{4.37}$$

(overbar for closure). We see that the projection P onto the overlapping four-dimensional S_i^*-invariant subspace $\mathcal{K} = \mathrm{span}\left\{1, z^{-1}, z^{-2}, z^{-3}\right\}$ commutes with the projection onto the first two subspaces. The respective products of P by these projections are

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{4.38}$$

and these two matrices span exactly the eigenspace of $\boldsymbol{\sigma}$ corresponding to eigenvalue 1. Also, each of the two subrepresentations has irreducible restriction to UHF_2, confirming the fact that the peripheral spectrum of $\boldsymbol{\sigma}$ consists of 1 alone.

4.1.2.2. *The case* $\theta = \frac{\pi}{2}$. When $\theta = \frac{\pi}{2}$,

$$a_0 = a_3 = 1/\sqrt{2}, \qquad a_1 = a_2 = 0, \tag{4.39}$$

so the associated low/high-pass filters and scaling relations are:

$$m_0(z) = \left(1 + z^3\right)/\sqrt{2}, \qquad \varphi(x/2) = \varphi(x) + \varphi(x-3), \tag{4.40}$$

$$m_1(z) = \left(1 - z^3\right)/\sqrt{2}, \qquad \psi(x/2) = \varphi(x) - \varphi(x-3). \tag{4.41}$$

See Figure 5 for the graphs of the corresponding φ and ψ. Applying the unitary $\frac{1}{\sqrt{2}}\left(\begin{smallmatrix}1 & 1\\ 1 & -1\end{smallmatrix}\right) \in \mathrm{U}(2)$ to this representation, we transform it into the representation with $m_0(z) = 1$, $m_1(z) = z^3$. We have already noted in (4.31) that the fixed point set of $\boldsymbol{\sigma}$ is three-dimensional in this case, and indeed, by [4, Proposition 8.2], this representation decomposes into 3 mutually disjoint irreducible representations given by restriction to the 3 subspaces

$$\begin{gathered}\overline{\operatorname{span}}\left\{z^{3n} \mid n = 0,1,2,\dots\right\}, \qquad \overline{\operatorname{span}}\left\{z^{3n} \mid n = -1,-2,\dots\right\}, \\ \overline{\operatorname{span}}\left\{z^{k} \mid k \text{ not divisible by } 3\right\}.\end{gathered} \tag{4.42}$$

The restriction to UHF_2 is still irreducible on the first two subspaces, while it decomposes into the two irreducible subrepresentations on

$$\overline{\operatorname{span}}\left\{z^{3k+1} \mid k \in \mathbb{Z}\right\}, \qquad \overline{\operatorname{span}}\left\{z^{3k+2} \mid k \in \mathbb{Z}\right\} \tag{4.43}$$

on the third subspace. Again the projection onto each of these subspaces commutes with P, and hence the eigenspace of $\boldsymbol{\sigma}$ corresponding to eigenvalue 1 is spanned by the three projections

$$\begin{pmatrix}1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, \quad \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&1\end{pmatrix}, \quad \text{and} \quad \begin{pmatrix}0&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&0\end{pmatrix}, \tag{4.44}$$

respectively, confirming that the eigenvalue 1 has multiplicity 3 in this case. Furthermore, if U is the unitary operator (2.21) on $\overline{\operatorname{span}}\left\{z^k \mid k \text{ not divisible by } 3\right\}$ that implements the gauge automorphism τ_{-1} there, we have $UT_i = -T_iU$. Hence

$$U\left(\xi\left(z^2\right)\right) = -\left(U\xi\right)\left(z^2\right) \quad \text{and} \quad U\left(z^3\xi\left(z^2\right)\right) = -z^3\left(U\xi\right)\left(z^2\right) \tag{4.45}$$

if ξ is in this subspace. This unitary U from (2.21) has to fix the two subspaces $\overline{\operatorname{span}}\left\{z^{3k+1} \mid k \in \mathbb{Z}\right\}$ and $\overline{\operatorname{span}}\left\{z^{3k+2} \mid k \in \mathbb{Z}\right\}$, and $U^2 = \mathbb{1}$, hence it is clear that

$$PUP = \pm\begin{pmatrix}0&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&0\end{pmatrix}. \tag{4.46}$$

It is easily verified directly that this is the eigenvector of $\boldsymbol{\sigma}$ corresponding to eigenvalue -1. This means that the group from (2.18)–(2.20) in this case is $\mathbb{Z}_2$, if $\mathcal{K}$ is taken to be $\operatorname{span}\{e_{-1}, e_{-2}\}$ and φ the trace state on $\mathcal{B}(\mathcal{K})$.

In conclusion we note that this $\mathcal{O}_2$-representation $\boldsymbol{\pi}^{(\theta)}$, $\theta = \frac{\pi}{2}$, as well as its restriction to UHF_2 has a decomposition into irreducibles which sets it apart from the other representations when $\theta \neq \frac{\pi}{2}$. We will see in the beginning of Section 4.1.2.5 that if $\theta = 0$, π, or $\frac{3\pi}{2}$, then the wavelet is still of Haar type, i.e., φ is of the form $\varphi = \chi_I$ where I is an interval of unit length. The position of the interval I varies (see Figure 4) in the three cases $\theta = 0$, π, or $\frac{3\pi}{2}$, while the mother function

$\psi^{(0)}$ is common for two of them, $\theta = 0, \pi$, and $\psi^{\left(\frac{3\pi}{2}\right)} = -\psi^{(0)} = -\psi^{(\pi)}$. All three satisfy $\psi(3-x) = -\psi(x)$. But, if $\theta = \frac{\pi}{2}$, then the nature of φ is somewhat different. From (4.40), we see that $\varphi\left(\frac{x}{2}\right) = \varphi(x) + \varphi(x-3)$; and by [18], then φ must have the form $\varphi = \frac{1}{3}\chi_S$ where S is a compact subset $\subset [0,3]$ with non-empty interior. It is determined by the identity

$$2S = S \cup (S+3) \tag{4.47}$$

(see Figure 5). It follows that $S = [0,3]$. In fact, iteration of (4.47) leads to the following representation which characterizes points x in S: $x = \sum_{k=1}^{\infty} d_k/2^k$, $d_k = 3\varepsilon_k$, $\varepsilon_k \in \{0,1\}$. Hence, using base 2 for the unit interval $[0,1]$, we get $S = [0,3]$. The derivation of $\varphi^{\left(\frac{\pi}{2}\right)}$ from the first Haar wavelet $\varphi^{(\pi)} = \chi_{[0,1]}$ is a special case of the substitution

$$m_0(z) \longmapsto m_0\left(z^3\right), \quad \text{or generally,} \quad m_0(z) \longmapsto m_0\left(z^{2p+1}\right). \tag{4.48}$$

If m_0 is an arbitrary low-pass filter with scaling function φ, then the argument from Remark 4.2 shows that $\widehat{\varphi_{2p+1}}(\omega) := \hat{\varphi}((2p+1)\omega)$ will determine the scaling function for the substitution $m_0\left(z^{2p+1}\right)$. Hence

(4.49)

$$\varphi_{2p+1}(x) = \frac{1}{2p+1}\varphi\left(\frac{x}{2p+1}\right), \quad \text{and} \quad \|\varphi_{2p+1}\|_{L^2(\mathbb{R})} = \frac{1}{\sqrt{2p+1}}\|\varphi\|_{L^2(\mathbb{R})}.$$

In our circular family, we have $m_0^{\left(\frac{\pi}{2}\right)}(z) = m_0^{(\pi)}\left(z^3\right)$. See further discussion in Section 4.1.2.4 and Remark 4.3.

4.1.2.3. *Intertwining of the cases $\theta = \frac{\pi}{2}$ and $\theta = \frac{3\pi}{2}$.* Let us summarize the description of these two representations. By (3.7)–(3.8) we have

$$S_0^{\frac{3\pi}{2}} e_n = \frac{1}{\sqrt{2}}(e_{1+2n} + e_{2+2n}), \qquad S_1^{\frac{3\pi}{2}} e_n = \frac{1}{\sqrt{2}}(-e_{1+2n} + e_{2+2n}), \tag{4.50}$$

and the irreducible invariant subspaces are

$$\mathcal{H}_+^{\frac{3\pi}{2}} = \overline{\operatorname{span}}\{e_{-1}, e_0, e_1, \dots\}, \qquad \mathcal{H}_-^{\frac{3\pi}{2}} = \overline{\operatorname{span}}\{e_{-2}, e_{-3}, \dots\}. \tag{4.51}$$

Similarly

$$S_0^{\frac{\pi}{2}} e_n = (e_{2n} + e_{3+2n})/\sqrt{2}, \qquad S_1^{\frac{\pi}{2}} e_n = (e_{2n} - e_{3+2n})/\sqrt{2}, \tag{4.52}$$

and the associated three irreducible invariant subspaces are

$$\begin{gathered} \mathcal{H}_+^{\frac{\pi}{2}} = \overline{\operatorname{span}}\{e_0, e_3, e_6, \dots\}, \qquad \mathcal{H}_-^{\frac{\pi}{2}} = \overline{\operatorname{span}}\{e_{-3}, e_{-6}, e_{-9}, \dots\}, \\ \mathcal{J} = \overline{\operatorname{span}}\{\dots, e_{-4}, e_{-2}, e_{-1}, e_1, e_2, e_4, \dots\}. \end{gathered} \tag{4.53}$$

We have noted that UHF_2 is not weakly dense in the last representation but it is so in the first four, so the last representation cannot be equivalent to any of the former four. Also the representation on $\mathcal{H}_+^{\theta}$ is disjoint from that on $\mathcal{H}_-^{\theta}$ by [4, Theorem 2.7], for $\theta = \frac{3\pi}{2}$ and for $\theta = \frac{\pi}{2}$. So the remaining possibility is that the representation on $\mathcal{H}_\pm^{\frac{3\pi}{2}}$ is unitarily equivalent to that on $\mathcal{H}_\pm^{\frac{\pi}{2}}$. Inspection of the expressions for $S_i^{\theta} e_n$ makes it plausible that the representation on $\mathcal{H}_+^{\frac{3\pi}{2}}$ is equivalent to that on $\mathcal{H}_-^{\frac{\pi}{2}}$, and that that on $\mathcal{H}_-^{\frac{3\pi}{2}}$ is equivalent to that on $\mathcal{H}_+^{\frac{\pi}{2}}$, and indeed, if one defines an isometry U by

$$U e_n = e_{-3n-6} \tag{4.54}$$

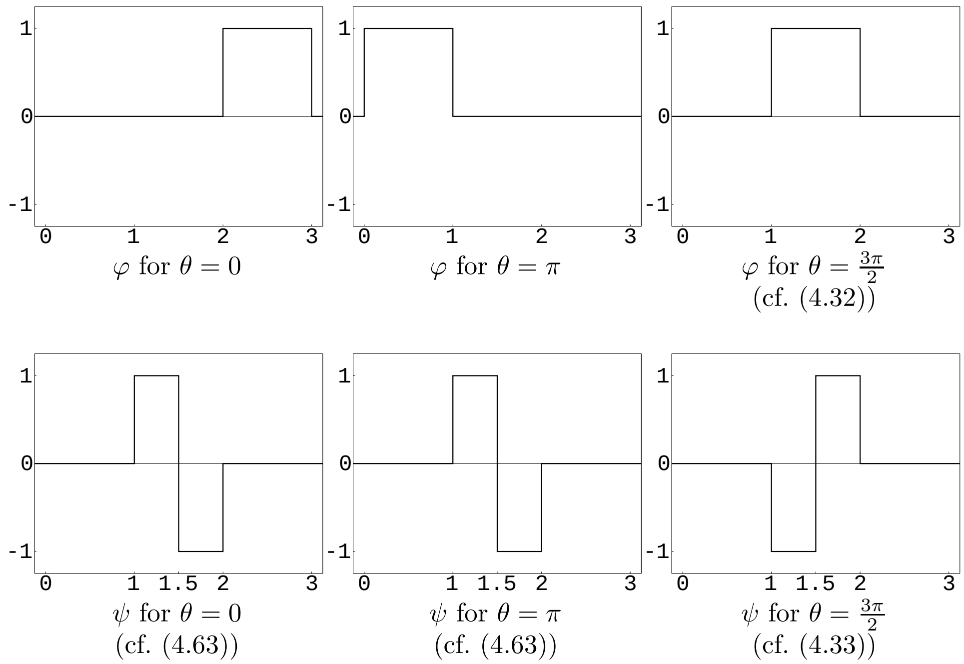


FIGURE 4. Father (φ) and mother (ψ) functions for θ equal to multiples of $\frac{\pi}{2}$: The symmetry $\psi(3-x) = -\psi(x)$

then $U|_{\mathcal{H}_+^{\frac{3\pi}{2}}}$ from $\mathcal{H}_+^{\frac{3\pi}{2}}$ to $\mathcal{H}_-^{\frac{\pi}{2}}$ and $U|_{\mathcal{H}_-^{\frac{3\pi}{2}}}$ from $\mathcal{H}_-^{\frac{3\pi}{2}}$ to $\mathcal{H}_+^{\frac{\pi}{2}}$ are unitary operators, and one computes

$$US_0^{\frac{3\pi}{2}} e_n = \left(e_{-6n-9} + e_{-6n-12}\right)/\sqrt{2} \qquad = S_0^{\frac{\pi}{2}} U e_n, \tag{4.55}$$

$$US_1^{\frac{3\pi}{2}} e_n = \left(-e_{-6n-9} + e_{-6n-12}\right)/\sqrt{2} \qquad = S_1^{\frac{\pi}{2}} U e_n. \tag{4.56}$$

Hence U intertwines the two representations, and if U is restricted to $\mathcal{H}_\pm^{\frac{3\pi}{2}}$ one obtains the expected unitary intertwiners

$$U_1 \colon \mathcal{H}_+^{\frac{3\pi}{2}} \longrightarrow \mathcal{H}_-^{\frac{\pi}{2}}, \qquad U_2 \colon \mathcal{H}_-^{\frac{3\pi}{2}} \longrightarrow \mathcal{H}_+^{\frac{\pi}{2}}. \tag{4.57}$$

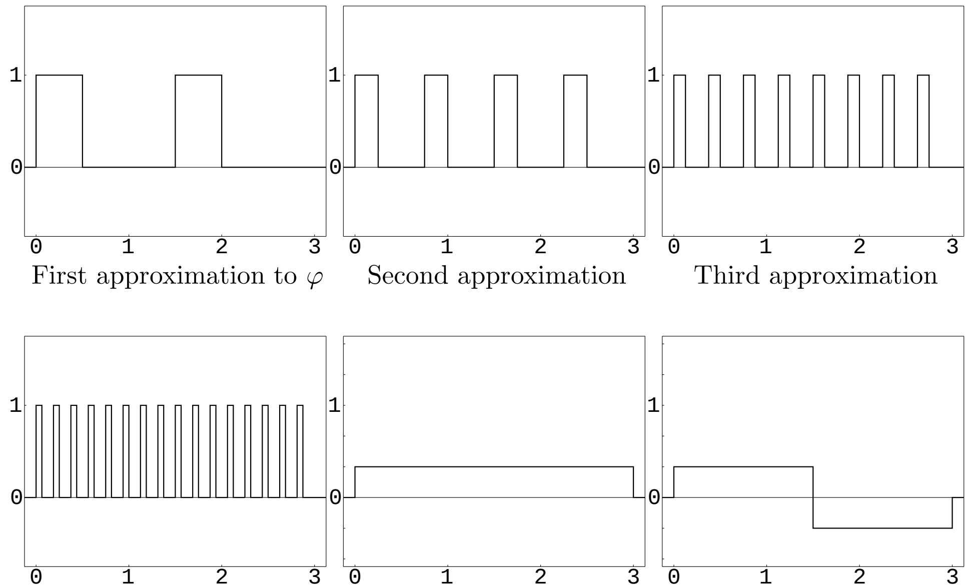
Now $U_1 e_{-1} = e_{-3}$, $U_2 e_{-2} = e_0$, and hence

$$P\left(xU_1 + yU_2\right)P = \begin{pmatrix} 0 & 0 & y & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & x & 0 & 0 \end{pmatrix} \tag{4.58}$$

for $x, y \in \mathbb{C}$, and this is exactly the fixed point set for the map

$$M_4 \ni A \longmapsto \sum_{i=0}^{1} V_i^{\frac{\pi}{2}} A V_i^{\frac{3\pi}{2}\,*}, \tag{4.59}$$

as it should be by (2.13)–(2.15). By [6, Theorem 5.1], these solutions A correspond to operators which intertwine the two representations.



First approximation to φ | Second approximation | Third approximation

Fourth approximation | Weak L^2 limit: φ for $\theta = \frac{\pi}{2}$ (cf. (4.40)) | ψ for $\theta = \frac{\pi}{2}$ (cf. (4.41))

FIGURE 5. Father (φ) and mother (ψ) functions for $\theta = \frac{\pi}{2}$, with cascade-algorithm approximations of father function φ: See discussion in Section 4.1.2.6

4.1.2.4. *Additional remarks on singular points and cycles.* Recall from [12, Theorem 6.3.6] and [21, Theorem 3.3.6] that in order that $\psi_{j,k}(x) = 2^{-\frac{j}{2}}\psi\left(2^{-j}x - k\right)$ shall be an orthonormal basis for $L^2(\mathbb{R})$ and not merely a tight frame, it is necessary and sufficient that the set

$$\left\{z \in \mathbb{T} \bigm| |m_0(z)| = \sqrt{2}\right\} = \{z \in \mathbb{T} \mid m_0(-z) = 0\} \tag{4.60}$$

does not contain a nontrivial cycle for the doubling map $z \mapsto z^2$, i.e., a finite cyclic subset unequal to $\{1\}$ invariant under the map $z \mapsto z^2$. Inspection of the polynomial $m_0^{(\theta)}(z)$ in (4.8) in the present case (4.24) reveals that the condition above is fulfilled for all $\theta \in \mathbb{T}$ with the sole exception

$$\theta = \pi/2, \tag{4.61}$$

where $m_0^{(\theta)}(z)$ is given by (4.40), and thus the set (4.60) consists of the three cube roots of 1. Indeed, the presence of a nontrivial cycle on $\mathbb{T}$ under $z \mapsto z^2$ would imply, by (4.60) and the fact that $m_0^{(\theta)}$ is a third-degree polynomial, that $m_0^{(\theta)}(\,\cdot\,)$ is a scalar multiple of $(z+1)\left(z + e^{i\frac{2\pi}{3}}\right)\left(z + e^{-i\frac{2\pi}{3}}\right) = z^3 + 1$, and this is precisely the case $\theta = \frac{\pi}{2}$ in (4.61). (See details of the argument below.) It is interesting that this is the case of Section 4.1.2.2 where the decomposition theory of the associated representation is most singular, being the only case where the restriction of one of the subrepresentations to UHF_2 has a nontrivial cyclic structure.

Let us give a more detailed justification of the statement above. First note that cycles on $\mathbb{T}$ are not subgroups of $\mathbb{T}$ but rather cyclic orbits on $\mathbb{T}$ under the $z \mapsto z^2$ action of one of the cyclic groups $\mathbb{Z}_k$, $k = 1, 2, \dots$. Such a cyclic orbit C_k with k distinct points $z_1, \dots, z_k$ must be of the form $z_1 \to z_2 \to \dots \to z_k \to z_1$, where $z_{i+1} = z_i^2$ if $i = 1, 2, \dots, k-1$, and $z_k^2 = z_1$. Hence points c in an orbit C_k must satisfy $c^{2^k} = c$, and each c must be a $\left(2^k - 1\right)$'th root of 1. Different orbits must be disjoint, and their union will be invariant under $z \mapsto z^2$ acting on $\mathbb{T}$. The converse is not true. For example, the subset $\{1, -1\} \subset \mathbb{T}$ is invariant under $z \mapsto z^2$ while not a cycle, and not even the union of cycles. Note also that we can have different $\left(2^k - 1\right)$'th roots c of 1 defining different cyclic orbits for the same k. If $k = 1$ or $k = 2$, then in each case there is only one orbit, but if $k = 3$, there are two choices. Since $m_0^{(\theta)}$ for each θ is a polynomial of degree 3, the cardinality of a cycle contained in the set (4.60) is at most 3. Thus, if z is contained in such a cycle, we must have one of the possibilities $z^2 = z$, $z^4 = z$, $z^8 = z$. Hence the cycles of length at most 3 are $\{1\}$, $\left\{\omega, \omega^2\right\}$ where $\omega = e^{i\frac{2\pi}{3}}$, $\left\{\rho, \rho^2, \rho^4\right\}$ where $\rho = e^{i\frac{2\pi}{7}}$, and $\left\{\bar{\rho}, \bar{\rho}^2, \bar{\rho}^4\right\} = \left\{\rho^6, \rho^5, \rho^3\right\}$. But as $m_0(-1) = 0$ always, $(z+1)$ is always a factor of $m_0(z)$, and since the cycle should be different from the trivial cycle $\{1\}$, we are reduced to the case $\left\{\omega, \omega^2\right\}$. The other cycles would make $m_0^{(\theta)}$ divisible by a polynomial of degree at least 4, which of course is impossible. Thus we are left with the case

$$m_0(z) = \frac{1}{\sqrt{2}} \prod_{k=0}^{2} \left(\omega^k + z\right) = \frac{1}{\sqrt{2}} \left(1 + z^3\right), \tag{4.62}$$

which is exactly the case $\theta = \frac{\pi}{2}$.

Note, more generally, that the wavelets which arise from substitutions, as defined in (4.48)–(4.49) with filter function $m_0^{(p)}(z) = m_0\left(z^{2p+1}\right)$, will have those additional cycles C_k which are contained in the $(2p+1)$'th roots of 1, $\left\{z \in \mathbb{T} \mid z^{2p+1} = 1\right\}$. We will show in a forthcoming paper that this leads to a decomposition of the representation of $\mathcal{O}_2$ associated to $m_0^{(p)}$ over the new cycles.

It is interesting to note that the same cycles as described above arise in a different context in [4] in connection with a family of discrete series of representations of $\mathcal{O}_N$. These representations are called permutative representations, and the cycles represent the finite decompositions of irreducible representations of $\mathcal{O}_N$ when restricted to UHF_N.

4.1.2.5. *The symmetry $\theta \mapsto \pi - \theta$.* Note that the two points $\theta = 0$ and $\theta = \pi$ are interesting in that the representation theory is regular, but these points correspond to mother and father functions which are simple rescalings of those of the Haar wavelet (see Figure 4):

(4.63)

$$\theta = 0\colon \begin{cases} m_0(z) &= \left(z^2 + z^3\right)/\sqrt{2} \\ m_1(z) &= (1 - z)/\sqrt{2} \end{cases} \qquad \theta = \pi\colon \begin{cases} m_0(z) &= (1 + z)/\sqrt{2} \\ m_1(z) &= \left(z^2 - z^3\right)/\sqrt{2} \end{cases}$$

Thus the representation of $\mathcal{O}_2$ is very sensitive to simple rescaling of φ and ψ. In fact the mother function ψ is the same in the two cases $\theta = 0$ and $\theta = \pi$, and this common ψ has the following symmetry property $\psi(3 - x) = -\psi(x)$, which in turn is a special case of a more general reflection symmetry (4.67) to be discussed in Proposition 4.1(a) below.

The symmetry $0 \mapsto \pi$ is a special case of a symmetry $\theta \mapsto \pi - \theta$, which we will now analyze further. If this transformation is substituted in (4.24), we note that it corresponds to the following reversal:

$$(a_0, a_2, a_2, a_3) \longmapsto (a_3, a_2, a_1, a_0)\,; \tag{4.64}$$

or equivalently,

$$m_0^{(\pi-\theta)}(z) = m_0^{(\theta)}\left(z^{-1}\right) z^3. \tag{4.65}$$

The following proposition shows that the $\theta \mapsto \pi - \theta$ reflection applied to $m_0^{(\theta)}$ implements the $x \mapsto 3 - x$ transformation on the scaling function φ (see Figure 3a,b). It is interesting to note that, despite this left-right mirror symmetry of the graphs in the family of scaling functions $\varphi^{(\theta)}$, the two associated representations of $\mathcal{O}_2$ on $L^2(\mathbb{T})$ which correspond, respectively, to θ and $\pi - \theta$, are *not* unitarily equivalent, by the results above, except of course at the two fixed points $\frac{\pi}{2}$ and $\frac{3\pi}{2}$ for $\theta \mapsto \pi - \theta$, where the representation theory also happens to be exceptional. See subsections 4.1.2.1 and 4.1.2.2 above.

Proposition 4.1. *Let $m_0^{(\theta)}(z)$ be the filter functions indexed by θ and corresponding to the given coefficients in the family* (4.24)*. Let $\varphi^{(\theta)}(x)$ be the associated scaling function* (*alias, father function*) *and $\psi^{(\theta)}$ the mother function corresponding to the pair $\left(m_0^{(\theta)}, m_1^{(\theta)}\right)$ of low/high-pass wavelet filters. Let $S_i^{(\theta)}$ be the corresponding operators from* (3.4).

(a) *The symmetry relations*

$$\varphi^{(\pi-\theta)}(x) = \varphi^{(\theta)}(3 - x)\,, \quad \theta \in [-\pi, \pi]\,,\ x \in \mathbb{R}, \tag{4.66}$$

$$\psi^{(\pi-\theta)}(x) = -\psi^{(\theta)}(3 - x)\,, \tag{4.67}$$

are valid.

(b) *The corresponding representations $\boldsymbol{\pi}^{(\theta)}$ and $\boldsymbol{\pi}^{(\pi-\theta)}$* (*given by $\boldsymbol{\pi}^{(\theta)}(s_i) = S_i^{(\theta)}$*) *satisfy*

$$W\boldsymbol{\pi}^{(\theta)} = \left(\boldsymbol{\pi}^{(\pi-\theta)} \circ \tau_{\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)}\right) W, \tag{4.68}$$

where $(Wf)(z) = z^{-3} f\left(z^{-1}\right)$, and $\tau_{\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)}$ is the automorphism of $\mathcal{O}_2$ given in (2.2) *for $g = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$.*

Proof. Introducing $z = e^{-i\omega}$, $\omega \in \mathbb{R}$, the identity (4.65) above reads

$$m_0^{(\pi-\theta)}(\omega) = e^{i3\omega} m_0^{(\theta)}(-\omega) = e^{i3\omega} \overline{m_0^{(\theta)}(\omega)}, \quad \omega \in \mathbb{R}. \tag{4.69}$$

Generally for third degree, the correspondence $m_0 \leftrightarrow \varphi$ is given by the following functional identity in $L^2(\mathbb{R})$:

$$\varphi(x/2)/\sqrt{2} = a_0\varphi(x) + a_1\varphi(x-1) + a_2\varphi(x-2) + a_3\varphi(x-3)\,, \tag{4.70}$$

and the boundary conditions, $\varphi(0) = \varphi(3) = 0$, i.e., φ is uniquely determined by these conditions and the normalization $\hat{\varphi}(0) = (2\pi)^{-\frac{1}{2}}$. See [27] for details. This applies to both the pair $\left(m_0^{(\theta)}, \varphi^{(\theta)}\right)$ and the pair $\left(m_0^{(\pi-\theta)}, \varphi^{(\pi-\theta)}\right)$, so we get

$$\begin{aligned} &\varphi^{(\pi-\theta)}(x/2)/\sqrt{2} \\ &= a_3\varphi^{(\pi-\theta)}(x) + a_2\varphi^{(\pi-\theta)}(x-1) + a_1\varphi^{(\pi-\theta)}(x-2) + a_0\varphi^{(\pi-\theta)}(x-3)\,. \end{aligned} \tag{4.71}$$

As noted, $\varphi^{(\pi-\theta)}(\,\cdot\,)$ is the unique normalized $L^2(\mathbb{R})$-solution to this identity, subject to $\varphi^{(\pi-\theta)}(0)=\varphi^{(\pi-\theta)}(3)=0$. But, if $\varphi^{(\theta)}$ is the solution corresponding to $m_0^{(\theta)}$, then a direct substitution $x\mapsto 6-x$ shows that the mirrored function $x\mapsto\varphi^{(\theta)}(3-x)$ satisfies (4.71), and we conclude from the uniqueness that

$$\varphi^{(\pi-\theta)}(x)=\varphi^{(\theta)}(3-x), \quad x\in\mathbb{R}, \tag{4.72}$$

as claimed in the Proposition. The proof of (4.67) is similar, or see Remark 4.2 below. We resume the proof of Proposition 4.1(b) after the following remark.

Remark 4.2. Proposition 4.1(a) may alternatively be proved from the Mallat algorithm as follows: If $\varphi^{(\theta)}$, $\psi^{(\theta)}$ are the father and mother functions at the angle θ, and the transformation $\theta\mapsto\pi-\theta$ is used on (4.24), we obtain (4.64) and (4.65) as before, i.e.,

$$m_0(z)\longmapsto z^3\overline{m_0(z)}=m_1(-z) \quad\text{and}\quad m_1(z)\longmapsto m_0(-z). \tag{4.73}$$

Applying the Mallat algorithm $\hat{\varphi}(t)=(2\pi)^{-\frac{1}{2}}\prod_{k=1}^{\infty}\left(m_0\left(e^{-it2^{-k}}\right)/\sqrt{2}\right)$, we obtain

$$\widehat{\varphi^{(\pi-\theta)}}(t)=e^{-i3t}\widehat{\varphi^{(\theta)}}(-t), \tag{4.74}$$

and thus by Fourier transform,

$$\varphi^{(\pi-\theta)}(x)=\varphi^{(\theta)}(3-x), \tag{4.75}$$

which is (4.66). On the other hand,

$$\psi(x)=\sqrt{2}\sum_k(-1)^k a_{3-k}\varphi(2x-k), \tag{4.76}$$

and so

$$\begin{aligned}\psi^{(\pi-\theta)}(x)&=\sqrt{2}\sum_k(-1)^k a_{3-k}^{(\pi-\theta)}\varphi^{(\pi-\theta)}(2x-k)\\&=\sqrt{2}\sum_k(-1)^k a_k^{(\theta)}\varphi^{(\theta)}(3-(2x-k))\\&=\sqrt{2}\sum_k(-1)^{3-k}a_{3-k}^{(\theta)}\varphi^{(\theta)}(2(3-x)-k)=-\psi^{(\theta)}(3-x),\end{aligned} \tag{4.77}$$

which is (4.67).

It is important to note that the infinite product argument works even if $\varphi^{(\theta)}(x)$ is not continuous in x. Since

$$\left|m_0^{(\theta)}(\omega)\right|^2+\left|m_0^{(\theta)}(\omega+\pi)\right|^2=2, \quad \omega\in\mathbb{R}, \tag{4.78}$$

it is known that the infinite products

$$(2\pi)^{-\frac{1}{2}}\prod_{k=1}^{\infty}2^{-\frac{1}{2}}m_0^{(\theta)}\left(\frac{\omega}{2^k}\right), \quad (2\pi)^{-\frac{1}{2}}\,2^{-\frac{1}{2}}m_1^{(\theta)}\left(\frac{\omega}{2}\right)\prod_{k=2}^{\infty}2^{-\frac{1}{2}}m_0^{(\theta)}\left(\frac{\omega}{2^k}\right) \tag{4.79}$$

are well defined and represent $\widehat{\varphi^{(\theta)}}$, $\widehat{\psi^{(\theta)}}$, where $\varphi^{(\theta)},\psi^{(\theta)}\in L^2(\mathbb{R})$ [12].

Proof of Proposition 4.1(b). Let us consider the two operators $S_0^{(\theta)}$ and $S_0^{(\pi-\theta)}$ in $L^2(\mathbb{T})$ individually, and as part of a pair of $\mathcal{O}_2$-representations. While the two $\mathcal{O}_2$-representations are inequivalent, the two S_0-operators alone are unitarily equivalent. This follows from the general fact that any operator of the form (1.9) coming from a wavelet is unitarily equivalent to the shift of infinite multiplicity by [5, Lemma 9.3]. The explicit intertwiner can also be calculated directly as follows: Let $m(z) = a_0 + a_1 z + \cdots + a_D z^D$, $m'(z) := z^D m\left(z^{-1}\right)$, and define three operators S, S', and W (acting on $f \in L^2(\mathbb{T})$) by

$$Sf(z) := m(z) f\left(z^2\right), \quad S'f(z) := m'(z) f\left(z^2\right), \quad Wf(z) := z^{-D} f\left(z^{-1}\right).$$

Then $W\colon L^2(\mathbb{T}) \to L^2(\mathbb{T})$ is a unitary intertwining operator for S and S', i.e.,

$$WS = S'W$$

holds, as can be verified by a direct calculation. The minus sign in the second symmetry formula (4.67) is still reflected in the $\mathcal{O}_2$-representations as follows. Let $D = 3$ and $m = m_0^{(\theta)}$, and consider the two $\mathcal{O}_2$-representations $\boldsymbol{\pi}^{(\theta)}$, $\boldsymbol{\pi}^{(\pi-\theta)}$, $i = 0, 1$. We then have $m_1^{(\pi-\theta)}(z) = -z^3 m_1^{(\theta)}\left(z^{-1}\right)$, and thus

$$WS_0^{(\theta)} = S_0^{(\pi-\theta)}W, \qquad WS_1^{(\theta)} = -S_1^{(\pi-\theta)}W, \tag{4.80}$$

where again $S_i^{(\theta)} = \boldsymbol{\pi}^{(\theta)}(s_i)$. Hence W intertwines the θ-representation $\boldsymbol{\pi}^{(\theta)}$ with the $(\pi-\theta)$-representation $\boldsymbol{\pi}^{(\pi-\theta)}$, modified by the automorphism of $\mathcal{O}_2$ induced by $g = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \in \mathrm{U}(2)$; see (2.2). ∎

4.1.2.6. *Continuity of scaling functions: Stability interval.* Historically the special case $\theta = \frac{7\pi}{6}$ in (4.24) was discovered first. In that case,

$$a_0 = \frac{1+\sqrt{3}}{4\sqrt{2}}, \quad a_1 = \frac{3+\sqrt{3}}{4\sqrt{2}}, \quad a_2 = \frac{3-\sqrt{3}}{4\sqrt{2}}, \quad a_3 = \frac{1-\sqrt{3}}{4\sqrt{2}},$$

which is the (by now) well known Daubechies wavelet; see Figure 3a,b [12, Chapter 6]. It was analyzed further in [27], where it was shown to have scaling function $\varphi(\,\cdot\,)$ continuous and one-sided differentiable in x, support on $[0,3]$, $\varphi(0) = \varphi(3) = 0$. It is left-differentiable at every dyadic x, but it is not right-differentiable at any dyadic x in $[0,3\rangle$. Since Daubechies established continuity by a matrix spectral estimate (see [12, Theorem 7.2.1]), it follows from her estimates that the scaling function $\varphi^{(\theta)}(x)$ will also be continuous in an open interval containing $\theta = \frac{7\pi}{6}$. It is interesting to note that Daubechies's spectral estimation involves the two matrices V_i^*, $i = 0, 1$, given in (4.25) above. The discussion in our previous section indicates that the stability interval in the θ variable, $\theta_0 < \theta < \theta_1$, must have $\pi < \theta_0$ and $\theta_1 < \frac{3\pi}{2}$.

The pictures of the scaling function in this paper are generated with the aid of the cascade algorithm described in [12, Section 6.5].

For uniform convergence of the cascade approximants to φ, one has to assume that φ is Hölder continuous [12, Proposition 6.5.2]. Figure 5 shows clearly that this uniform convergence may fail abysmally even when φ is a simple step function. However, we see from Figure 5 that the cascade approximants converge in the distribution sense, and even in the weak-L^2 sense, to φ when $\theta = \frac{\pi}{2}$.

The other assumption in Daubechies's cascade approximation is the orthogonality of $\mathbb{Z}$-translates, in the form [12, (6.5.4)–(6.5.5), p. 204], and, as we will discuss

in Remark 4.3 below, that fails when $\theta = \frac{\pi}{2}$, but is satisfied at all other values of θ by Section 4.1.2.4 above.

More importantly, Daubechies states in [12, Chapter 6 footnote 9 and Section 6.3] that, even if φ is not assumed continuous, we still have $L^2(\mathbb{R})$ norm convergence of the cascade-algorithm approximation, as long as the $\mathbb{Z}$-translates are mutually orthogonal; and, as we noted, this orthogonality holds whenever $\theta \neq \frac{\pi}{2}$. This will be discussed in a forthcoming paper [3].

Remark 4.3. Since $\varphi^{\left(\frac{\pi}{2}\right)} = \frac{1}{3}\chi_{[0,3]}$, it is geometrically clear that the $\mathbb{Z}$-translates of $\varphi^{\left(\frac{\pi}{2}\right)}$ in $L^2(\mathbb{R})$ will not be mutually orthogonal (see Figure 5), and we have shown in Section 4.1.2.4 that $\varphi^{\left(\frac{\pi}{2}\right)}$ is the unique scaling function in the family $\left\{\varphi^{(\theta)}\right\}$ which does not have orthogonal $\mathbb{Z}$-translates. The cascade algorithm, which is used in generating the present graphics, is based on an iteration of (4.18) but is also closely connected to iteration of F_0^* in (1.12). Let

$$c_k := \frac{1}{\sqrt{2}} \int_{\mathbb{R}} \overline{\varphi(x-k)}\varphi\left(\frac{x}{2}\right) dx. \tag{4.81}$$

In the case when $\{\varphi(\,\cdot\,-k)\}_{k\in\mathbb{Z}}$ is an orthonormal basis, we get $c_k = a_k$, $k \in \mathbb{Z}$, by (4.18); but, in general, we have a discrepancy $c_k \neq a_k$ which leads to a rather poor approximation with a_k-cascades. For a more explicit estimate we need the following:

Lemma 4.4. *Let m_0 be a low-pass wavelet filter with corresponding scaling function φ and suppose that the $\mathbb{Z}$-translates of φ are orthogonal. Let φ_p be the scaling function corresponding to the substitution $m_0\left(z^{2p+1}\right)$, and let*

$$c_k^{(p)} := \frac{1}{\sqrt{2}} \int_{\mathbb{R}} \overline{\varphi_p(x-k)}\varphi_p\left(\frac{x}{2}\right) dx.$$

Then

$$\sum_k \left|c_k^{(p)}\right|^2 \leq \frac{1}{2p+1}. \tag{4.82}$$

Proof. From [12] or [5, Proposition 12.4], we have

$$\sum_{l\in\mathbb{Z}} \left|\hat{\varphi}(\omega + 2\pi l)\right|^2 \equiv \frac{1}{2\pi}. \tag{4.83}$$

Since $\widehat{\varphi_p}(\omega) = \hat{\varphi}((2p+1)\omega)$, we conclude that

$$\sum_l \left|\hat{\varphi}((2p+1)(\omega + 2\pi l))\right|^2 \leq \frac{1}{2\pi}. \tag{4.84}$$

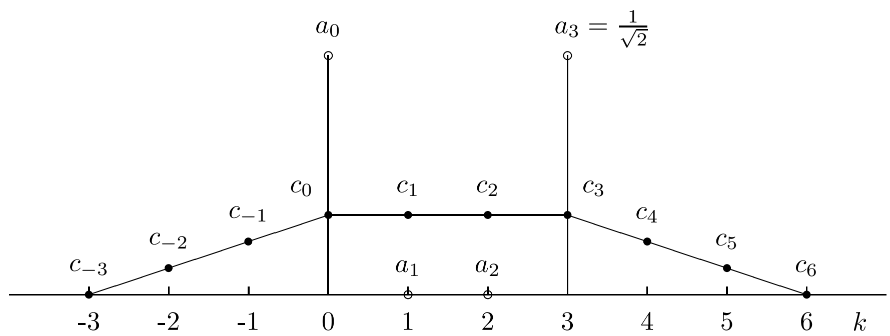


FIGURE 6. Correlation coefficients for $\theta = \frac{\pi}{2}$: The correlation coefficients c_k, for which $\sum_{k=-2}^{5} c_k^2 = \frac{23}{81} < \frac{1}{3}$, as compared to the scaling coefficients a_k, for which $\sum_{k=0}^{3} a_k^2 = 1$.

This second summation is just one of the $2p+1$ residue classes for the full $\mathbb{Z}$ summation in (4.83). But the formula for c_k yields

$$\begin{aligned}
\sum_{k\in\mathbb{Z}} \left| c_k^{(p)} \right|^2 &= \sum_{k\in\mathbb{Z}} \left| \int_0^{2\pi} e^{ik\omega} m_0\left((2p+1)\,\omega\right) \sum_{l\in\mathbb{Z}} \left|\hat{\varphi}\left((2p+1)\left(\omega+2\pi l\right)\right)\right|^2 \, d\omega \right|^2 \\
&= 2\pi \int_0^{2\pi} \left| m_0\left((2p+1)\,\omega\right)\right|^2 \left(\sum_{l} \left|\hat{\varphi}\left((2p+1)\left(\omega+2\pi l\right)\right)\right|^2 \right)^2 d\omega \\
&= \frac{1}{(2p+1)\,2\pi} \int_0^{2\pi} \left|m_0\left(\omega\right)\right|^2 \sum_{j=0}^{2p} \left(\sum_{l\in\mathbb{Z}} 2\pi \left|\hat{\varphi}\left(\omega + \left(j+(2p+1)\,l\right)2\pi\right)\right|^2 \right)^2 d\omega \\
&\leq \frac{1}{(2p+1)\,2\pi} \int_0^{2\pi} \left|m_0\left(\omega\right)\right|^2 \sum_{j=0}^{2p} \sum_{l\in\mathbb{Z}} 2\pi \left|\hat{\varphi}\left(\omega + \underbrace{\left(j+(2p+1)\,l\right)}_{n}2\pi\right)\right|^2 d\omega \\
&= \frac{1}{(2p+1)\,2\pi} \int_0^{2\pi} \left|m_0\left(\omega\right)\right|^2 \underbrace{\sum_{n\in\mathbb{Z}} 2\pi \left|\hat{\varphi}\left(\omega + n\cdot 2\pi\right)\right|^2}_{1} d\omega \\
&= \frac{1}{2p+1}\cdot\frac{1}{2\pi} \int_0^{2\pi} \left|m_0\left(\omega\right)\right|^2 d\omega = \sum_{k\in\mathbb{Z}} \left|a_k\right|^2 \frac{1}{2p+1} = \frac{1}{2p+1}. \qquad \square
\end{aligned}$$

We now illustrate this for $\varphi^{\left(\frac{\pi}{2}\right)}$. Since $\varphi^{\left(\frac{\pi}{2}\right)} = \frac{1}{3}\chi_{[0,3]}$, it is easy to compute exactly the correlation coefficients c_k of (4.81). The nonzero coefficients are:

$$c_{-2} = c_5 = 1/9\sqrt{2}, \quad c_{-1} = c_4 = 2/9\sqrt{2}, \quad \text{and} \quad c_0 = c_1 = c_2 = c_3 = 1/3\sqrt{2},$$

which should be compared with (4.39). They are also illustrated in Figure 6, and a comparison with Figure 5 suggests that replacing the a_k's in the cascades with the c_k's might possibly lead to a better approximation. Good approximations are not known in the non-orthogonal case. For more details, see [12, pp. 204–206].

The c_k numbers are those which may be inserted into the spline approximation that is also discussed in [12, pp. 206–207] to build in tight frame parameters in the approximation.

The problem with this substitution of the c_k's into the cascade algorithm is that, in the non-orthogonal case, we will have (see Lemma 4.4) $\sum_k |c_k|^2 < 1$. Compare this to the normalization property $\sum_k |a_k|^2 = 1$ from (4.3), or (4.10) in the special case.

5. Conclusions

We have demonstrated how a representation-theoretic approach to the construction of compactly supported wavelets in $\mathbb{R}^d$ leads to:

(i) a coordinate-free display of the examples,
(ii) a finite-dimensional matrix algorithm for computing irreducibility properties,
(iii) a formula for decomposition into orthogonal sums of irreducibles.

The theory is illustrated in the simplest cases where the power of the representation-theoretic approach comes into play.

Acknowledgements. We are indebted to Rune Kleveland for assistance with the computations in Section 4.1.2, to Erik Løw for informing us about the cascade algorithm used in the graphics, to the University of Oslo for support, and to Brian Treadway for excellent typesetting, graphics, discovering the symmetry $\theta \mapsto \pi - \theta$, and manuscript coordination.

References

1. Christoph Bandt, *Self-similar sets* 5: *Integer matrices and fractal tilings of* $\mathbb{R}^n$, Proc. Amer. Math. Soc. **112** (1991), 549–562.
2. ______, *Self-similar tilings and patterns described by mappings*, The Mathematics of Long-Range Aperiodic Order (Waterloo, Ontario, 1995) (R. Moody, ed.), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., vol. 489, Kluwer Academic Publishers Group, 1997.
3. O. Bratteli and P.E.T. Jorgensen, *Convergence of the cascade algorithm at irregular scaling functions*, in preparation.
4. ______, *Iterated function systems and permutation representations of the Cuntz algebra*, Mem. Amer. Math. Soc., to appear.
5. ______, *Isometries, shifts, Cuntz algebras and multiresolution wavelet analysis of scale N*, Integral Equations Operator Theory **28** (1997), 382–443.
6. O. Bratteli, P.E.T. Jorgensen, A. Kishimoto, and R. Werner, *Pure states on* $\mathcal{O}_d$, submitted to Journal of Operator Theory.
7. O. Bratteli and D.W. Robinson, *Operator Algebras and Quantum Statistical Mechanics*, 2nd ed., vol. I, Springer-Verlag, Berlin–New York, 1987.
8. ______, *Operator Algebras and Quantum Statistical Mechanics*, 2nd ed., vol. II, Springer-Verlag, Berlin–New York, 1996.
9. A. Cohen and R.D. Ryan, *Wavelets and Multiscale Signal Processing*, Applied Mathematics and Mathematical Computation, vol. 11, Chapman & Hall, London, 1995.
10. J. Cuntz, *Simple* C^**-algebras generated by isometries*, Comm. Math. Phys. **57** (1977), 173–185.
11. Stephan Dahlke, Wolfgang Dahmen, and Vera Latour, *Smooth refinable functions and wavelets obtained by convolution products*, Appl. Comput. Harmon. Anal. **2** (1995), 68–84.
12. I. Daubechies, *Ten Lectures on Wavelets*, CBMS-NSF Regional Conf. Ser. in Appl. Math., vol. 61, Society for Industrial and Applied Mathematics, Philadelphia, 1992.
13. Ken R. Davidson and David R. Pitts, *The algebraic structure of non-commutative analytic Toeplitz algebras*, Math. Ann., to appear.
14. J. Dixmier, *Traces sur les* C^**-algèbres,* II, Bull. Sci. Math. (2) **88** (1964), 39–57.
15. ______, *Les* C^**-algèbres et Leurs Représentations*, Gauthier-Villars, Paris, 1969.

16. D.E. Evans, *On $\mathcal{O}_N$*, Publ. Res. Inst. Math. Sci. **16** (1980), 915–927.
17. D.E. Evans and Y. Kawahigashi, *Quantum Symmetries on Operator Algebras*, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 1998.
18. K. Gröchenig and W.R. Madych, *Multiresolution analysis, Haar bases, and self-similar tilings of* $\mathbf{R}^n$, IEEE Trans. Inform. Theory **38** (1992), 556–568.
19. P.N. Heller, H.L. Resnikoff, and R.O. Wells, Jr., *Wavelet matrices and the representation of discrete functions*, Wavelets: A Tutorial in Theory and Applications (C.K. Chui, ed.), Wavelet Anal. Appl., vol. 2, Academic Press, Boston, 1992, pp. 15–50.
20. P.N. Heller and R.O. Wells, Jr., *The spectral theory of multiresolution operators and applications*, Wavelets: Theory, Algorithms, and Applications (Taormina, 1993) (C.K. Chui, L. Montefusco, and L. Puccio, eds.), Wavelet Anal. Appl., vol. 5, Academic Press, San Diego, 1994, pp. 13–31.
21. L. Hörmander, *Lectures on harmonic analysis*, Dept. of Mathematics, Box 118, S-22100 Lund, 1995.
22. P.E.T. Jorgensen, L.M. Schmitt, and R.F. Werner, *q-canonical commutation relations and stability of the Cuntz algebra*, Pacific J. Math. **165** (1994), 131–151.
23. S.G. Mallat, *Multiresolution approximations and wavelet orthonormal bases of* $\mathbf{L}^2(\mathbf{R})$, Trans. Amer. Math. Soc. **315** (1989), 69–87.
24. K. McCormick and R.O. Wells, Jr., *Wavelet calculus and finite difference operators*, Math. Comp. **63** (1994), 155–173.
25. D. Pollen, *Parametrization of compactly supported wavelets*, company report, AWARE, Inc., AD890503.1.4, 1989.
26. ———, $\mathrm{SU}_I(2, F[z, 1/z])$ *for F a subfield of* $\mathbf{C}$, J. Amer. Math. Soc. **3** (1990), 611–624.
27. ———, *Daubechies' scaling function on* $[0, 3]$, Wavelets: A Tutorial in Theory and Applications (C.K. Chui, ed.), Wavelet Anal. Appl., vol. 2, Academic Press, Boston, 1992, pp. 3–13.
28. H.L. Resnikoff and R.O. Wells, Jr., *Wavelet analysis and the geometry of Euclidean domains*, J. Geom. Phys. **8** (1992), 273–282.
29. R.S. Strichartz, *Self-similarity in harmonic analysis*, J. Fourier Anal. Appl. **1** (1994), 1–37.
30. R.O. Wells, Jr., *Parametrizing smooth compactly supported wavelets*, Trans. Amer. Math. Soc. **338** (1993), 919–931.
31. M.V. Wickerhauser, *Best-adapted wavelet packet bases*, Different perspectives on wavelets (San Antonio, TX, 1993) (I. Daubechies, ed.), Proc. Sympos. Appl. Math., vol. 47, Amer. Math. Soc., Providence, RI, 1993, pp. 155–171.

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